THE INSTITUTE FOR ENQUIRING MINDS

redefining
access
resetting
attitudes

enquiringminds.org
info@enquiringminds.org
+61 448 218652

Mathematical Methods Units 3/4

Calculus

Practice Questions

- Differentiation
- Applications of Differentiation
- Integration
- Applications of Integration

Short Answer Questions

Question 1

Determine the derivative of $y = \dfrac{2\sin(x)}{x^2}$

Question 2

If $f(x) = \dfrac{\log_e(2x-1)}{1-x}$, determine the exact value of $f'(2)$.

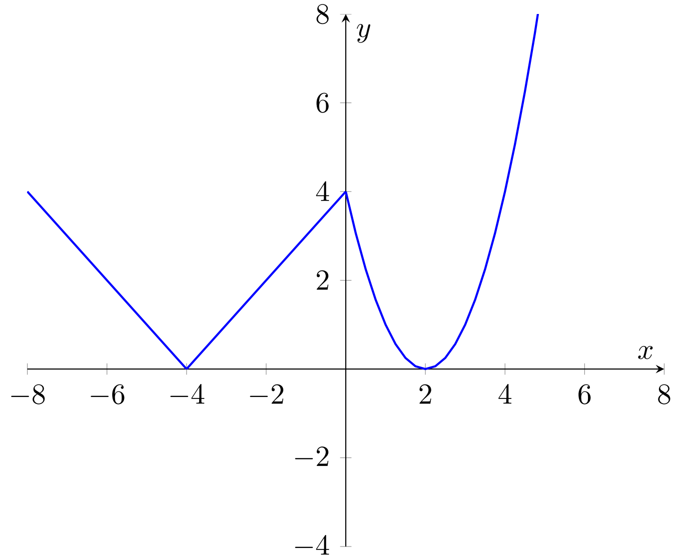
Question 3

a) Differentiate $y = e^{\sin(3x)}$ with respect to x.

b) For $f(x) = \sin(2x)e^{3x}$, find the exact value of $f'(\frac{\pi}{2})$.

Question 4

a) For the graph on the diagram below, sketch on the same set of axes the graph of the derivative function.



b) Write down the domain of the derivative function.

Question 5

a) $\frac{d}{dx}(\frac{\cos{(x^2)}}{x})$

b) $\int \frac{3}{(4-2x)^2} + 3 \; dx$

Question 6

A block of ice is melting such that its volume, V cubic centimetres at any time, t is given by

$$V = 100 - 25t - 0.01t^2, t \in (0, 35)$$

a) Find the exact average rate of change, in cubic centimetres per minute, of the volume over the first ten minutes.

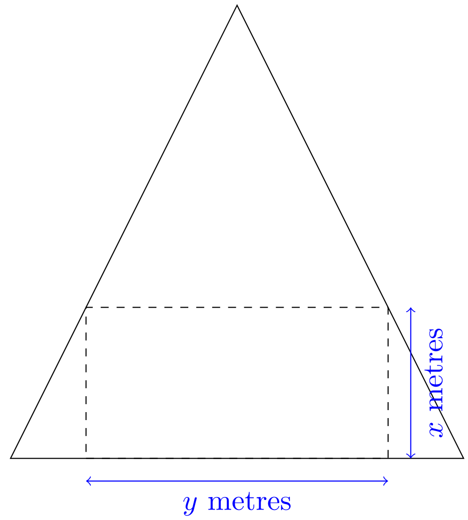
b) Find the exact instantaneous rate of change, in cubic centimetres per minute, of the volume when $t = 10$, correct to one decimal place.

Question 7

Adrenaline levels experienced by a circus performer are found to follow the rule $a = \dfrac{50}{t^2 - 2t + 2}$, where a is the adrenaline level, t minutes after a dangerous stunt begins. Find the time when the adrenaline level is maximum.

Question 8

A rectangular window is placed inside an isosceles triangle section of a roof. The base of the triangle is 2m and the other sides are 3m.



a) Find the exact value of the height of the triangle.

b) If the height of the rectangle is x and its width is y, express y in terms of x.

c) Hence state the area of the rectangle in terms of x.

d) Using calculus find (as an exact value) the maximum possible area of the window.

Question 9

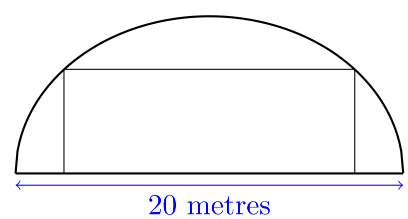
Given that $\int_1^5 h(x)\ dx = 4$, find the value of $\int_5^1 2(h(x) - 1)\ dx$

Question 10

An arched bridge over a road has a semi-circular shape. The width of the road is 20m.

A wide rectangular load needs to fit through the arch.

Find to one decimal place the dimensions of the rectangle corresponding to the largest cross-sectional area that can fit under the bridge.



Question 11

a) Use calculus to differentiate $y = x\cos(x)$

b) Hence, determine the exact value of $\displaystyle\int_0^{\pi} x\sin(x)\ dx$.

Question 12

The displacement of a particle from a fixed point is given by $x(t) = -t^2(t-5), 0 \leq t \leq 4$

a) Find rules for the velocity, $v(t)$, and acceleration, $a(t)$, in terms of t, where t is the time in seconds and x is the displacement in millimetres.

b) Over what domains are $v(t)$ and $a(t)$ are defined.

c) Find the velocity of the particle after 3.5 seconds.

d) Find the exact time when the particle is a maximum distance the fixed point.

e) Find the exact time when the particle is at maximum velocity.

Question 13

A piece of wire 1 metre line is cut into two pieces, one of which is used to form a square and the other, a circle. Describe the circle and square that would be formed to produce the least total area. What is this area ?

Question 14

A rocket soars into space with a velocity equation modelled by $v(t) = -1000e^{-0.5t} + 1000$ in kilometres per hour.

a) What is the velocity of the rocket one hour after take-off, to the nearest kilometre per hour ?

b) Use calculus to determine the acceleration of the rocket 1 hour after take-off to the nearest kilometre per hour.

c) Use calculus to determine the displacement of the rocket after 1 hour to the nearest metre.

Question 15

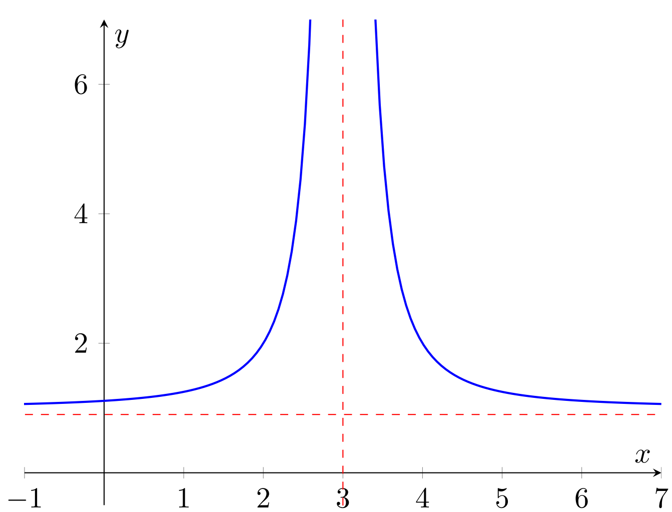
Find $f(x)$, if $f'(x) = 3\sin(2x)$ and $f(\pi) = -1$.

Question 16

Find the exact area bounded by the lines: $y = -\frac{1}{2}e^{-2x}$, the x axis, $x = 1$ and $x = 2$.

Question 17

For $\mathbb{R}\backslash\{3\} \to \mathbb{R}$, $f(x) = \frac{1}{(x-3)^2} + 1$.



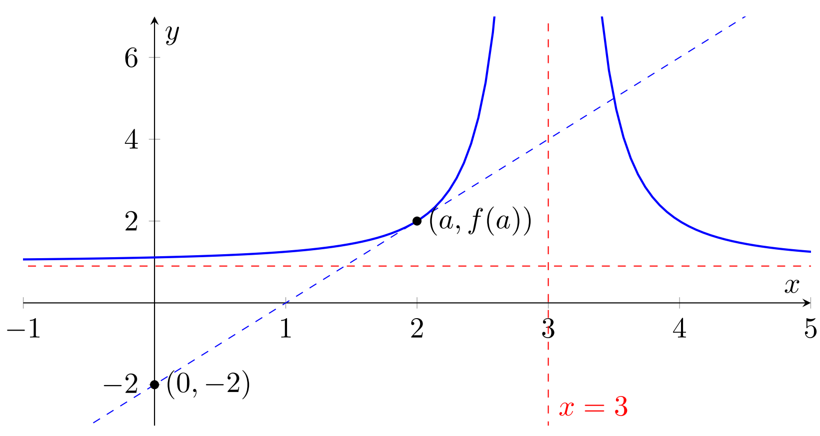
a) **i.** Over what interval is $f(x)$ differentiable.

ii. Find $f'(x)$

iii. What is the range of $f'(x)$

iv. Using the result from part **iii.**, explain why $f(x)$ has no stationary points.

b) The tangent to the curve of $f(x)$ at point $(a, f(a))$ has a y-intercept of (0,-2). The tangent line has been drawn onto the curve as shown below.



i. Find $f(a)$ and $f'(a)$ in terms of a.

ii. Use $\dfrac{y_2 - y_1}{x_2 - x_1}$ to write down an expression for the gradient of the tangent, in terms of a.

iii. Hence, find the exact value of a and $f(a)$.

iv. Hence, find the rules of the tangent and normal to the curve at the point $(a, f(a))$.

Question 18

A right cone sits on top of a cylinder. The radius of the cylinder is r cm and the height of the cylinder is h cm. The cone is the same height as the cylinder.

a) Find an expression for the total volume V cm^3,in terms of r and h.

b) Show that the surface area S cm^2 is given by $S = 2\pi rh + \pi r^2 + \pi r\sqrt{h^2 + r^2}$.

c) If the value of $V = 1000\pi$

i. Find h in terms of r

ii. Find S in terms of r

iii. Find $\frac{dS}{dr}$ and solve the equation $\frac{dS}{dr} = 0$ for r, correct to two decimal places.

iv. Sketch the graph of S against r. Give the minimum value of S and the value of r for which this occurs, correct to two decimal places.

Question 19

Given that $\int_0^5 h(x)\ dx = 3$ and $\int_0^2 2h(x) = -2$, calculate $\int_5^2 h(x)\ dx$.

Question 20

a) Find the derivative of $2\log_e(\sin(x))$.

b) Hence, evaluate $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos(x)}{\sin(x)}\ dx$, expressing your answer in the form $\log_e(a)$.

Question 21

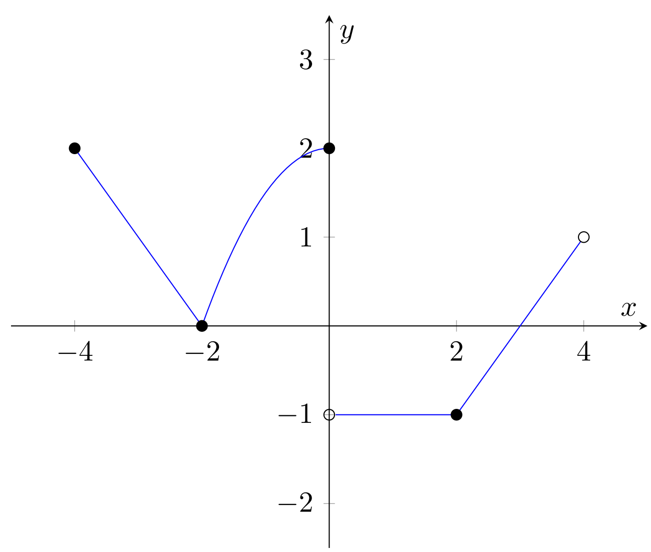
John is riding from school and has a velocity of $4t^2 - 3t$ metres per minute where t represents the time in minutes that he rides, after leaving school. Let x represent John's displacement in an easterly direction from a fixed point. After 10 minutes John's displacement is 4 km west.

a) How far is his school from this fixed point.

b) What is John's displacement, from the fixed point after 20 minutes.

Question 22

The graph of $g(x)$, which has domain [-4,4) is shown on the left:



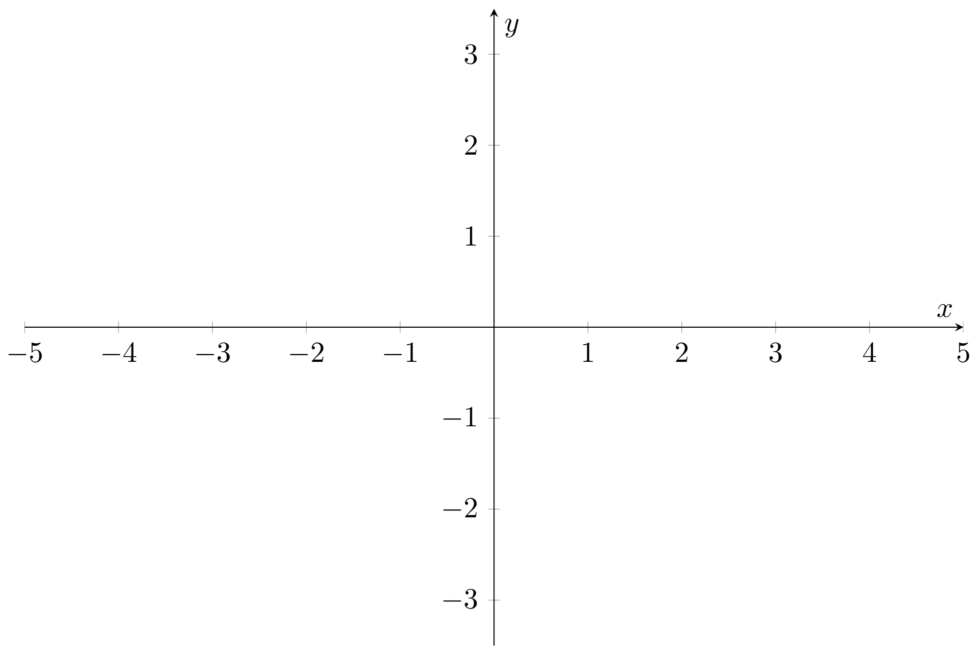
a) State the value of each limit, If a limit does not exist, write 'does not exist.'

i. $\lim_{x \to 0} g(x)$

ii. $\lim_{x \to 2} g(x)$

b) State the value(s) of x for which the graph of $g(x)$ is discontinuous.

c) On the pair of axes below, sketch the graph of $g'(x)$.



d) State the domain of $g'(x)$.

Question 23

The rate of decrease of an insect population can be given by

$$\frac{dI}{dt} = Ae^{-kt}$$

where t represents the number of years after the beginning of 2000 and I represents the number of insects.

a) At the start of 2000, the rate of decrease was 20000 insects per year. State the value of A

b) By the beginning of 2002, the rate of decrease was 15000 per year. Find k in the form $k = p\log_e\left(\frac{b}{c}\right)$, where $p, b, c > 0$

c) This model became ineffective at the beginning of 2004, the new model becomes

$$\frac{dI}{dx} = -22000e^{-2x}$$

where x represents the number of years since the beginning of 2004.
At the beginning of 2004 there were 11250 insects.

i. Find the equation $I(x)$ which represents the number of insects in the colony (from 2004)

ii. Using this model during which year is it predicted that there will be less than 300 insects.

Question 24

Let the function f be defined on the set of real numbers where:

$$f'(x) = 2 - \cos(x) \text{ and } f(\pi) = 0.$$

a) Find $f(x)$

b) Explain why f has no stationary points.

c) Find the minimum value of the gradient of f.

Question 25

Let $g(x) = \log_e(2x^2 - 3)$

a) Find $g'(x)$

b) Hence, evaluate $\int_2^3 \frac{8x}{2x^2 - 3}\ dx$. Give your answer in simplest form.

Question 26

a) Given $\int_0^6 f(x)\ dx = 0$and $\int_3^6 f(x)\ dx = 2$,

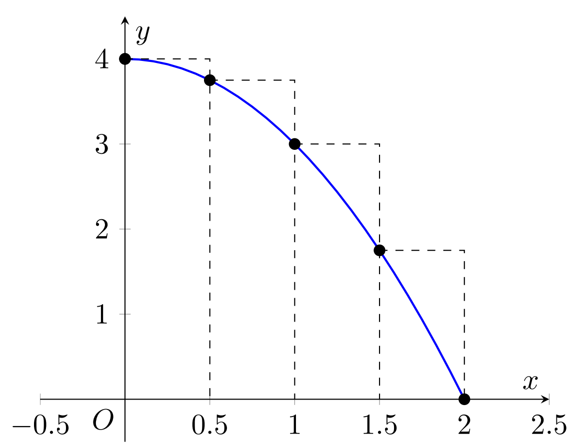
i. Find $\int_0^3 f(x)\ dx$.

ii. Evaluate $\int_3^6 \frac{4(f(x)) + 2}{3}\ dx$

b) If $\int_2^4 g(x)\ dx = \left[15ax^3 + \frac{3}{7}x^2\right]_2^4$ and $\int_4^2 g(x)\ dx = 4$, Find the exact value of a.

Question 27

Using 4 rectangles, calculate the left endpoint estimate approximation for the area under the curve $y = -x^2 + 4$ between $x = 0$ and $x = 2$.
Leave your answer as a simplified fraction.

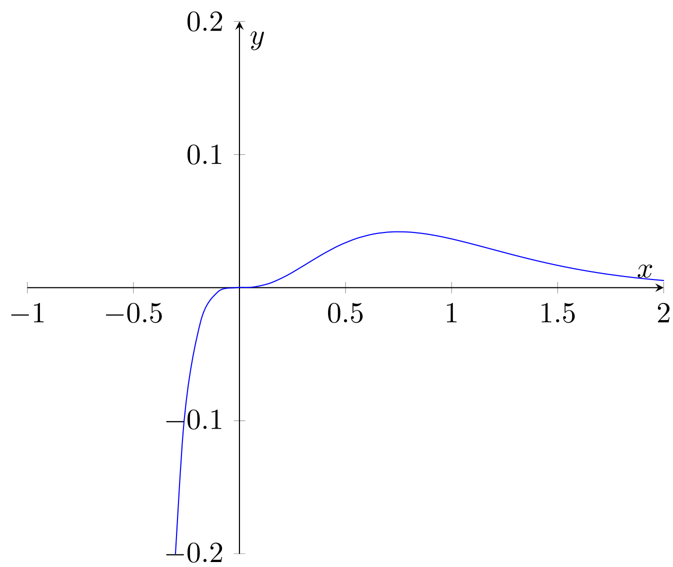


Question 28

Consider the function $f : \mathbb{R} \to \mathbb{R}, f(x) = 2x^3e^{-4x}$.

a) If the derivative of this function is written in the form $f'(x) = e^{-4x}(ax^2 + bx^3)$ evaluate a and b.

The graph $y = f(x)$ is as shown.



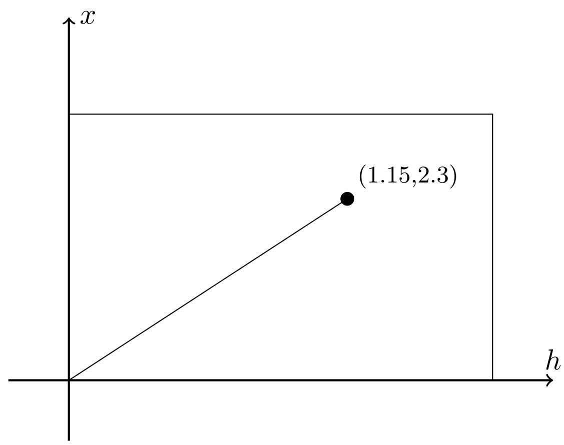
b) Find the exact coordinates of two stationary points and state their nature. (You must also show the nature of the stationary points.)

c) Find the equation of the tangent to the curve at the point where $x = 1$.

d) Using the two upper rectangles find the area enclosed by the curve, the lines $x = 1$ and $x = 2$.

Question 29

A skateboarder rides his skateboard up a ramp as shown below. The distance x (in metres) represents the horizontal displacement of the front of the skateboard from the base of the ramp and h is the vertical height from the ground (in metres).



The rate of change of the height of the skateboard with respect to x, $\frac{dh}{dx}$ after reaching the top of the ramp is approximated by $\frac{dh}{dx} = \frac{-80x}{7} + \frac{60}{7}, x \in [1.15, k]$ where $k > 1.15$.

a) Given that $h = 2.3$m when $x = 1.15$m, find h in terms of x.

b) At what horizontal distance,k, does the front of the skateboard reach the ground? (Correct to 2 decimal places)

c) The equation that models the increasing height against the horizontal distance when the board is on the ramp is modelled by $p(x) = 2x, x \in [0, 1.15)$
Find the area enclosed by the functions $p(x), h(x)$ and the x-axis.

Question 30

a) Differentiate $y = (x+2)\cos(2x)$.

b) Hence, find the anti-derivative of $(x+2)\sin(2x)$.

Question 31

a) Given that $f(x) = \dfrac{2x+1}{x-5}$ can be written as $f(x) = 2 + \dfrac{11}{x-5}$, find $\displaystyle\int \frac{2x+1}{x-5}dx$.

b) Find the equation of the tangent to the curve $y = f(x)$ at $x = 1$.

Question 32

Differentiate $y = e^{1-x^2}$ and hence find the exact value of $\int_0^4 xe^{1-x^2}\,dx$.

Question 33

Suppose an enclosure is constructed in the shape of a sector with radius r and angle x (measured in radians) between the two radii.

a) **i.** Find the perimeter of the enclosure in terms of r and x.

ii. Find the area of the enclosure in terms of r and x.

b) The enclosure is constructed (still in the shape of a sector) with $20m$ of fencing.

i. Show that the enclosed area is given by $A = (10 - r)r$.

ii. What is the maximum area (in m^2) that can be enclosed by this 20m fence.

Question 34

If $\int_0^4 g(x)dx = 10$ and $\int_4^5 g(x)dx = 1$. Calculate $\int_0^5 \frac{(g(x)-3)}{2}dx$

Question 35

Given that the derivative of $\log_e(\cos(2x))$ is $-2\tan(2x)$, calculate $\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} 4\tan(2x)dx$, giving your answer in the form $\log_e B$ where $B \in \mathbb{R}^+$.

Question 36

For the function $g(x) = 3x + 2\cos(x)$

a) Find the derivative of $g(x)$

b) Hence explain why $g(x)$ is an increasing function for all values of x.

Question 37

Let $g : \mathbb{R}^+ \to \mathbb{R}, g(x) = x - \log_e x$.
The equation of a tangent to the graph of g is $y = -\dfrac{x}{2} + k$.
Find the exact value of k.

Question 38

Find $\displaystyle\int \left(\frac{x + e^x}{xe^x}\right)\, dx$ where $x > 0$.

Question 39

Consider the function $f(x) = xe^{kx} - 2$ where k is a constant.

a) Find $f(0)$

b) Find $f'(x)$.

c) If the turning point occurs when $x = -\dfrac{1}{2}$, find the value of k.

d) Use your answer to part b. to find $\displaystyle\int xe^{5x}\,dx$

Question 40

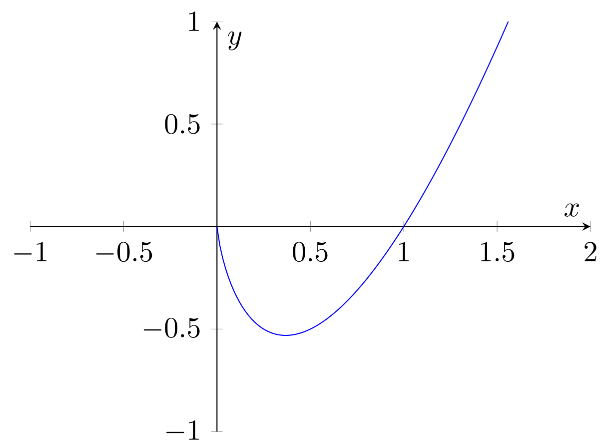
Let $f : \mathbb{R} \to \mathbb{R}, f(x) = x^2 + k$, where k is a real number.

a) Determine the equation of the tangent to the graph of f at the point where $x = a$.

b) Hence show that k cannot be negative if this tangent is to pass through the origin.

Question 41

The following graph is of $f(x) = x \log_2(x)$
Use the right endpoint method to approximate the definite integral $\int_0^2 x \log_2(x)\, dx$ using intervals of width 0.5.
Give your answer in the form $a + b \log_2(c)$, where a, b and c are rational numbers.



Question 42

Given $\dfrac{d}{dx}(e^x \sin(2x)) = e^x \sin(2x) + e^x 2\cos(2x)$,

Find $\displaystyle\int_0^{\frac{\pi}{2}} e^x(2\cos(2x) + \sin(2x) - 1)\ dx$.

Question 43

The tangent to the graph $f(x) = -x^2 + 6x$ at $x = a$ passes through (0,9)
Find the possible values of a.

Question 44

A normal to the curve of $y = \sqrt{2x-3}$ has equation $y = -x + a$, where a is a real constant, Find the value of a.

Question 45

a) State the derivative of $\log_e(x^2+2)$.

b) Hence, use calculus to find the value of a and b if $\displaystyle\int_1^2 (x + \frac{x}{x^2+2})\,dx = \log_e(a) + b$.

Question 46

An infection is transferred at such a rate that the number of people N, who are not infected can be modelled by $\dfrac{dN}{dt} = -0.16t$, where t is the number of days after the first diagnosis.
In a school of 800 students, how many students will not be infected after 20 days ?

Question 47

Find the point on the curve $y = \sqrt{x}$ which is closest to point (1,0).

Question 48

a) If $\displaystyle\int_1^4 f(x)\ dx = 2$ then $\displaystyle\int_1^4 2f(x) + 4\ dx$ is equal to:

b) If $\displaystyle\int_0^a f(x)\ dx = 4$ then $\displaystyle\int_0^{7a} f(\frac{x}{7}) - 9\ dx$ is equal to:

Question 49

a) Find the derivative of $\sin(x)\log_e(\sin(x))$

b) Hence use this result to evaluate $\int \cos(x)\log_e(\sin(x))\ dx$

Question 50

The table below shows some value of two functions, $f(x)$ and $g(x)$, and their derivatives $f'(x)$ and $g'(x)$.

x	1	2	3	4
$f(x)$	5	4	-1	3
$g(x)$	1	-2	2	-5
$f'(x)$	5	6	0	7
$g'(x)$	-6	-4	-3	4

Calculate the following

a) $\frac{d}{dx}(f(x)+g(x))$, when $x=4$

b) $\int_1^3 (g'(x)+6)\ dx$

Question 51

A medical team located at point A is directly across a river from point B. They must get an injured person to point D (on the shore of the river, on the same side as B) as quickly as possible. The team can row a boat across the river to point B and run along the shore to point D, or can row to some intermediate point C along the river bank and then run to point D. If the team can row at 6 km/h and run at 10km/h, where on the opposite shore should they land their boat. The river is 4km wide.

Question 52

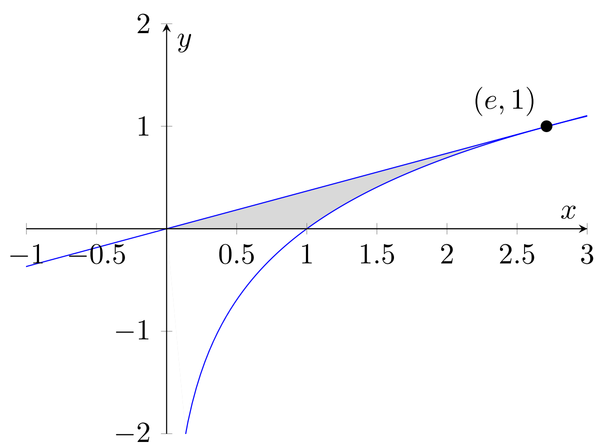
Someone with 750m of fencing wants to enclose a rectangular area and then divide it into four pens with fencing parallel to one side of the rectangle.
What is the largest possible total area of the four pens.

Question 53

a) Find the equation of the tangent line to the curve $y = \log_e(x)$ at the point $(e, 1)$ and verify that the origin is on this line.

b) Show that $\dfrac{d}{dx}(x\log_e(x) - x) = \log_e(x)$

c) The diagram shows the region enclosed by the curve $y = \log_e(x)$, the tangent line in part **a.**, and the line $y = 0$.
Use the result from part **b.** to find the area of the shaded region.



Question 54

A jeep is in a desert at a point P located 40km from a point Q, which lies on a long straight road. The driver can travel at 45km/h on the desert and 75km/h on the road. The driver will win a prize if he arrives at the finish line at point F, on the road, 50km from Q, in 84 minutes or less. What route should he travel to minimise the time of travel? Does he win the prize?

Question 55

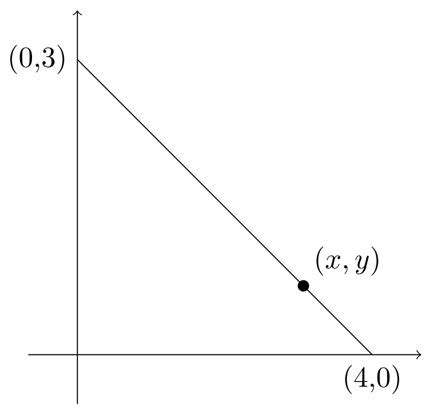
Given that $\dfrac{dy}{dx} = \dfrac{6}{(1-2x)^2}$ and $y = 2$ when $x = 1$. Find an equation for y in terms of x.

Question 56

Find all points on the circle $x^2 + y^2 = a^2$ such that the product of the x-coordinate and the y-coordinate is as large as possible.

Question 57

Find the coordinates of P that maximises the area of the rectangle shown below.



Question 58

Given that $\dfrac{d}{dx}(x\sin(x)) = 2x\cos(2x) + \sin(2x)$

a) Hence determine an anti-derivative of $x\cos(2x)$

b) Use your result from **a.** to show that $\displaystyle\int_0^{\frac{\pi}{4}} x\cos(2x)\;dx = \frac{\pi}{8} - \frac{1}{4}$.

Question 59

Consider the function $f(x) = \frac{1}{2}x^2$ and $g(x) = \frac{x^4}{16} - \frac{1}{2}x^2$ on the domain [0,4].

a) Use CAS to sketch both functions over [0,4]

b) Write the vertical distance d between the functions as a function of x and use calculus to find the value of x for which d is maximum.

c) Find the equation of the tangent lines to the graphs of f and g at the critical number found in part **b.** Graph the tangent lines, what is the relationship between the lines ?

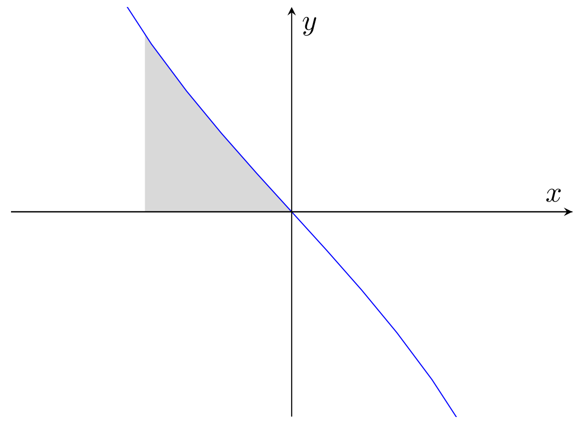
Question 60

Given that $\frac{d}{dx}(2x\cos(x)) = 2\cos(x) - 2x\sin(x)$, find $\int_0^{\pi} x\sin(x)\ dx$.

Question 61

Consider the shaded region which is bounded by $x = -\frac{\pi}{6}, y = -3\tan(x)$ and the x axis.

a) Write down that definite integral that would resolve to the area of the shaded area.

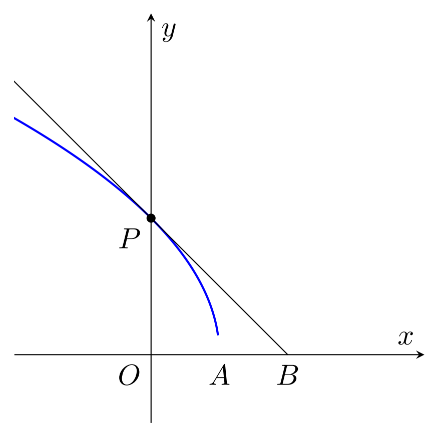


b) If $f(x) = \log_e(\cos(x))$ where $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$

c) Hence calculate the area of the shaded region, giving your answer in the form $B\log_e(A)$ square units.

Question 62

The graph of the function $f : (-\infty, 1] \to \mathbb{R}, f(x) = k\sqrt{1-x}$ where k is a positive integer, is shown below.
Point P is the y-intercept of the graph of f. The tangent to the graph f at point P has its x-intercept at point B as shown.



a) Find the equation of the tangent to the graph of f at point P, in terms of k.

b) Show that B is the point (2,0).

c) The shaded area in the diagram is enclosed by the graph of f, the tangent at point P and the x-axis. Find, in terms of k, the area of theis shaded region.

The normal to the graph of f at point P intersects with the line $x = 2$ at point C.

d) The area of $\triangle BCP$, expressed in terms of k, is given by $A(k)$
Show that $A(k) = k + \dfrac{4}{k}$.

e) Find $A'(k)$

f) Find the minimum area, in square units, of $\triangle BCP$.

Question 63

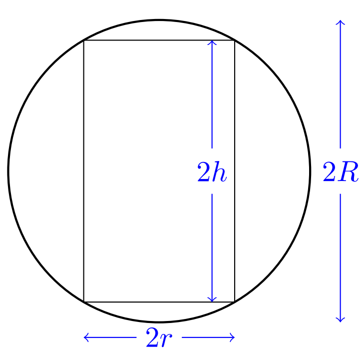
If $\int_0^4 h(x)\ dx = 7$, evaluate $\int_0^{16} h(\frac{x}{4})\ dx$.

Question 64

A cone is carved from a sphere, with the vertex of the cone at the center of the sphere. If the sphere is of radius 1m, find the vvolume of the largest possible cone.

Question 65

Find the dimension of the right circular cylinder of the largest volumne that can be inscribed in a sphere of radius R.



Question 66

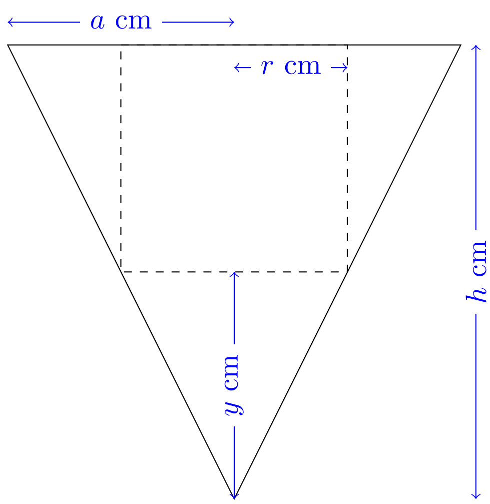
The bottom of a 8m high mural painted on a vertical wall is 13m above the ground. The lens of a camera fixed to a tripod is 4m above the ground.
How far from the wall should the camera be placed to photograph the mural with the largest possible angle.

Question 67

The base of a triangle is on the x-axis, one side lies along the line $y = 3x$, and the third side passes through the point (1,1). What is the slope of the third side if the area of the triangle is to be minimum.

Question 68

A right circular cylinder of volume V cm ^{3} and radius r is inscribed in a right circular cone with height h cm and base radius a cm, the two figures having a common axis of symmetry, where a and h are constants. $0 < r < a$.



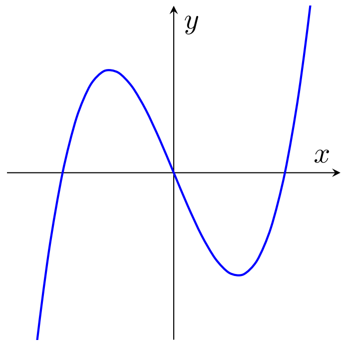
a) Express V in terms of r, a and h.

b) Find the maximum value of V.

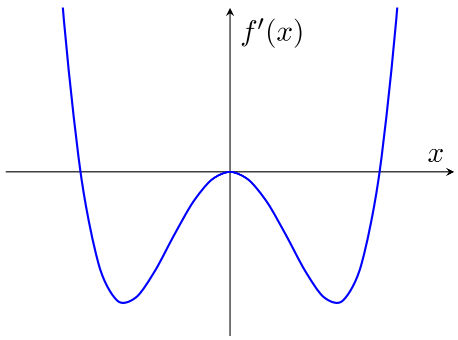
Multiple Choice Questions

Question 1

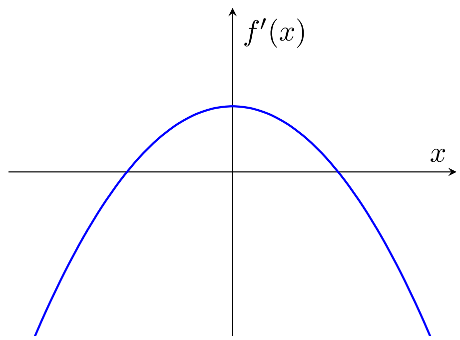
The graph of the function $y = f(x)$ is shown below.



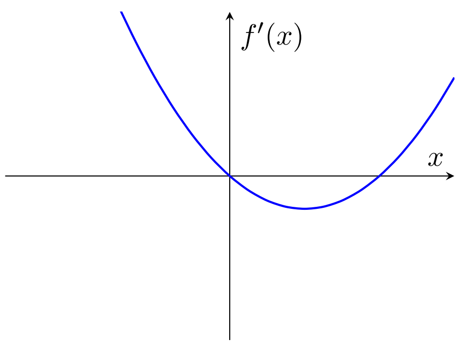
A.



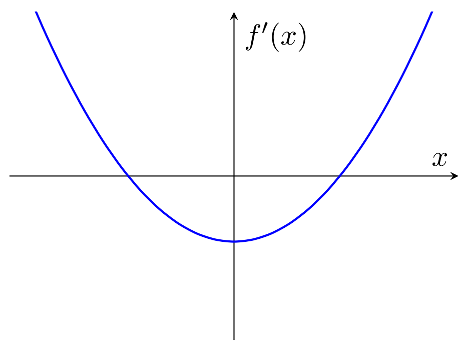
B.



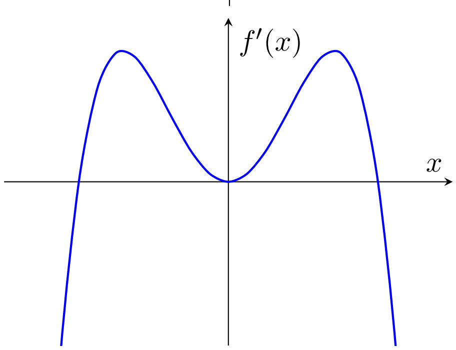
C.



D.



E.



Question 2

The derivative of $g(x)e^x$ is:

A. $g'(x)e^x$

B. $e^x(g'(x) + g(x))$

C. $g'(x)e^x + g'(x)e^x$

D. $xg(x)e^1$

E. $xg'(x)e^1$

Question 3

The derivative of $\log_e(\sin(2x))$ is:

A. $\tan(2x)$

B. $\dfrac{2}{2\tan(2x)}$

C. $-2\tan(2x)$

D. $\dfrac{2\sin(2x)}{\cos(2x)}$

E. $\dfrac{1}{\cos(2x)}$

Question 4

The graph with the rules: $f(x) = \begin{cases} 6, & x < 0 \\ x^2 + 2, & 0 \le x \le 2 \\ \dfrac{1}{x}, & x > 2 \end{cases}$

A. $x = 0$ and $x = 2$

B. $x = 0$ only

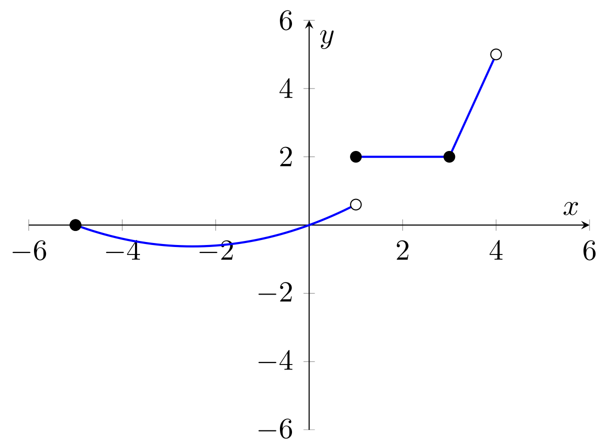
C. $x = 2$ only

D. $x = \dfrac{1}{2}$ only

E. $x = 0$ and $x = 6$

Question 5

The graph of the hybrid function $y = h(x)$ is shown below.



Hence $h(x)$ is:

A. Not differentiable at $x = 1$, but is differentiable at $x = -5, x = 3$ and $x = 4$

B. Not differentiable at $x = 1$ and $x = 4$, but is differentiable at $x = -5$ and $x = 3$

C. Not differentiable at $x = 1, x = 3$ and $x = 4$, but is differentiable at $x = -5$

D. Not differentiable at $x = -5, x = 1$ and $x = 4$, but is differentiable at $x = 3$

E. Not differentiable at $x = -5, x = 1, x = 3$ and $x = 4$

Question 6

The derivative $\dfrac{e^{4x}}{3\cos(x)}$ is:

A. $\dfrac{12e^{4x}\cos(x) - 3e^{4x}\sin(x)}{9\cos^2(x)}$

B. $\dfrac{3e^{4x}\sin(x) - 12e^{4x}\cos(x)}{9\cos^2(x)}$

C. $\dfrac{3e^{4x}\cos(x) - 12e^{4x}\sin(x)}{9\cos^2(x)}$

D. $\dfrac{12e^{4x}\cos(x) + 3e^{4x}\sin(x)}{9\cos^2(x)}$

E. $\dfrac{3e^{4x}\cos(x) + 12e^{4x}\sin(x)}{9\cos^2(x)}$

Question 7

Let $f(x)$ and $g(x)$ be functions such that $f(-1) = -3, f'(-1) = 2, g(-1) = -1$ and $g'(-1) = 5$.
If $h(x) = \dfrac{g(x)}{f(x)}$, then the value of $h'(-1)$ is equal to:

A. $\dfrac{5}{2}$

B. $-\dfrac{17}{9}$

C. $\dfrac{17}{9}$

D. $\dfrac{13}{9}$

E. $-\dfrac{13}{9}$

Question 8

If $\displaystyle\int_1^3 f(x)\ dx = 3$, then $\displaystyle\int_1^3 (3f(x) + 2)\ dx$ is equal to:

A. 9

B. 10

C. 11

D. 12

E. 13

Question 9

The area enclosed by the graph of $y = 1 - e^{x+3}$, the x-axis and the y-axis is given by:

A. $\int_{-3}^{0} (e^{(x+3)} - 1)\ dx$

B. $\int_{-3}^{0} (1 - e^{(x+3)})\ dx$

C. $\int_{-3}^{0} (e^{(x+3)} + 1)\ dx$

D. $\int_{-3}^{e^3-1} (e^{(x+3)} - 1)\ dx$

E. $\int_{-3}^{e^3-1} (1 - e^{(x+3)})\ dx$

Question 10

What is the derivative of $\dfrac{(e^x - e^{2x})^2}{e^{2x}}$

A. $-2e^x(1 + e^x)$

B. $2e^x(1 - e^x)$

C. $1 - 2e^x + e^{2x}$

D. $1 - 2e^x + 2e^{2x}$

E. $-2e^x(1 - e^x)$

Question 11

The exact value of $h'(2)$ if $h(x) = (x - 3)(3e^{2x} - e^x)$ is:

A. $3e^4 - 2e^2$

B. $3e^4$

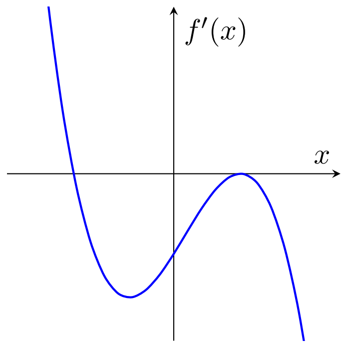
C. $-3e^4$

D. $3e^4 + 2e^2$

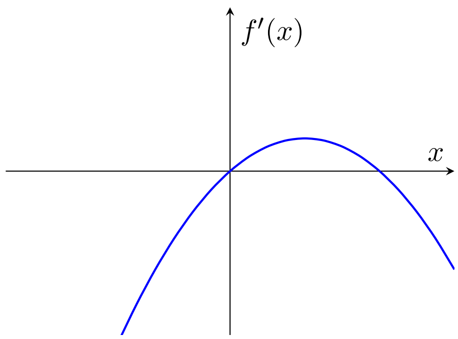
E. $2e^2 - 3e^4$

Question 12

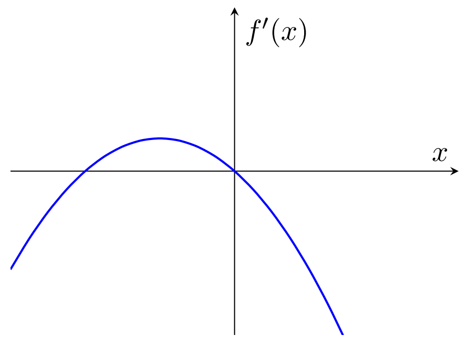
The graph of the function $y = f(x)$ is shown below.



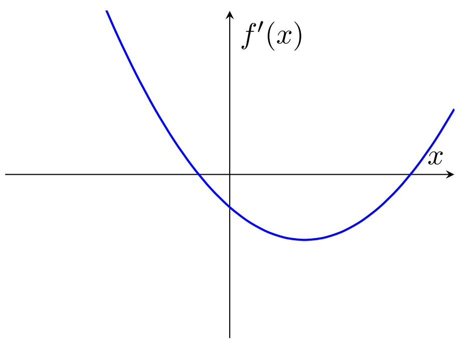
A.



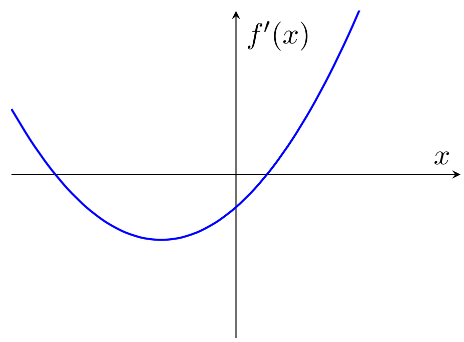
B.



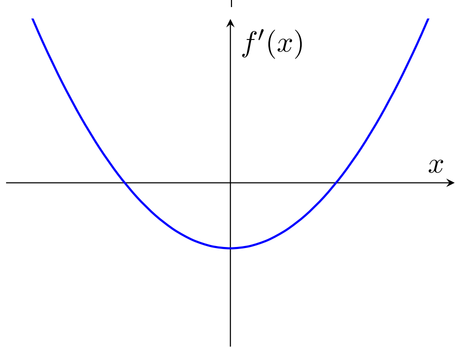
C.



D.



E.



Question 13

Let $f(x) = 2e^{g(3x)}$. Therefore $f'(x)$ is equal to:

A. $6e^{g(3x)}g'(x)$

B. $3^{g(3x)}g'(3x)$

C. $6^{g'(3x)}$

D. $6e^{g(3x)}g'(3x)$

E. $2e^{g(3x)}g'(3x)$

Question 14

If $\dfrac{dy}{dx} = \dfrac{2}{(4x+1)^{\frac{3}{2}}}$, where $(4x+1) > 0$ then y is:

A. $\dfrac{-12}{(4x+1)^{\frac{5}{2}}} + c$

B. $\dfrac{-1}{8(4x+1)^2} + c$

C. $\dfrac{x^2}{4} + \dfrac{x}{2} + c$

D. $\dfrac{1}{2}\log_e(4x+1) + c$

E. $\dfrac{-1}{(4x+1)^{\frac{1}{2}}} + c$

Question 15

The gradient of the function $f : \mathbb{R} \to \mathbb{R}, f(x) = \dfrac{x}{5-x^2}$ is positive for:

A. $x \in \mathbb{R}\backslash\{\pm\sqrt{5}\}$

B. $-\sqrt{5} < x < \sqrt{5}$

C. $x \in \mathbb{R}$

D. $x > -\sqrt{5}$

E. $x < 0$

0.1 Question 16

The rate of flow of water into a tank is given by $\frac{dV}{dt} = 5(10-t)e^{-(t+2)}, 0 \leq t \leq 10$, where V is the amount of water in the tank in litres and t is the time in minutes. Initially the tank is empty. Find the amount of water in the tank when t=10 minutes.

A. $45e^{-2} + 5e^{-12}$ litres

B. $5e^{-12}$ litres

C. $45e^{-2}$ litres

D. $45e^{-12} + 5e^{-2}$ litres

E. 0 litres

Question 17

A graph of the derivative function $y = f'(x)$ is shown. Which one of the following statements is true?

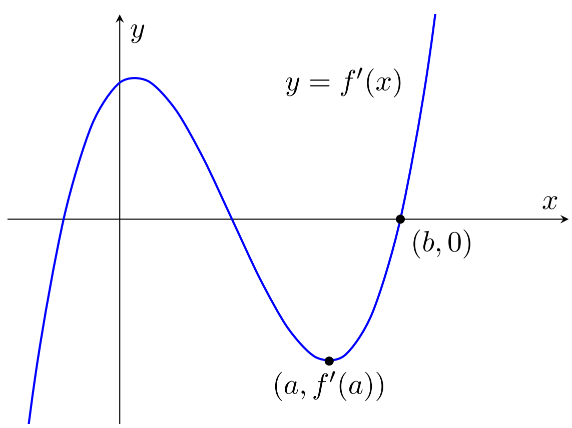
A. The graph of $y = f(x)$ is increasing over the domain $[a, b]$.

B. The graph of $y = f(x)$ has exactly two stationary points.

C. The graph of $y = f(x)$ could have a point of inflection at $x = b$.

D. The graph of $y = f(x)$ could be a graph in the form $y = ax^4 + dx^3 + cx^2 + dx + e$

E. It is not possible to determine what type of stationary points are on the graph $y = f(x)$ from the information provided.



Question 18

The equation of the normal to the curve with equation $y = x^2 f(x)$ at (3,9), given $f(3) = 2$ and $f'(x) = -2$ would be:

A. $y - 9 = 6(x - 3)$

B. $y - 9 = 2(x - 3)$

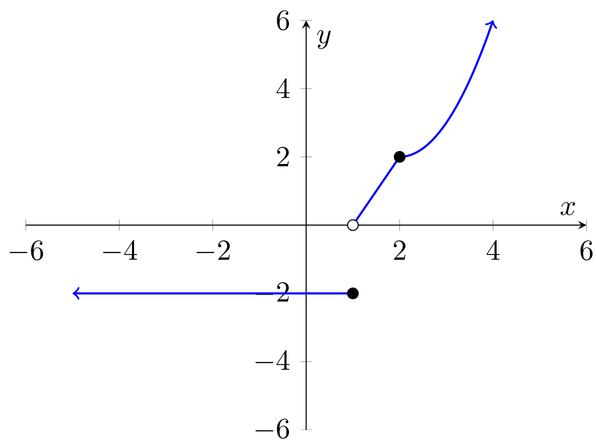
C. $y - 9 = \frac{1}{2}(x - 3)$

D. $y - 9 = -\frac{1}{6}(x - 3)$

E. $y - 9 = \frac{1}{6}(x - 3)$

Question 19

For the hybrid function shown, which one of the following statements are true



A. $y = f(x)$ is continuous at all points.

B. $y = f(x)$ is differentiable over the interval $(-\infty, a]$

C. For all $x \in (a, b)$, where $a < b, f(a) > f(b)$.

D. The function $y = f(x)$ can be defined with exactly 2 rules

E. $y = f(x)$ would have a gradient of zero at the coordinates $(a - 2, f(a - 2))$.

Question 20

The left endpoint approximation of the definite integral $\int_1^4 \log_e(x)$ using intervals of width 1 is:

A. $\log_e(1.5)$

B. $\log_e(5)$

C. $\log_e(6)$

D. $2\log_e(2)$

E. $3\log_e(2)$

Question 21

Suppose $y = \dfrac{f(x)}{x}$ and $f(3) = 5$ and $f'(3) = -2$, then when $x = 3$. $\dfrac{dy}{dx} =$

A. $-\dfrac{1}{9}$

B. $-\dfrac{1}{3}$

C. $-\dfrac{2}{3}$

D. $-\dfrac{11}{9}$

E. $-\dfrac{11}{3}$

Question 22

If $g(x) = \pi \tan(f(x))$ then $g'(x)$ would be equal to:

A. $\dfrac{\pi}{\cos^2(f'(x))}$

B. $\tan(f(x)) + \dfrac{\pi f'(x)}{\cos^2(f(x))}$

C. $\dfrac{\pi}{\cos^2(f(x))}$

D. $\dfrac{\pi f'(x)}{\cos(f(x))^2}$

E. $\dfrac{\pi f'(x)}{\cos^2(f(x))}$

Question 23

If the graph with equation $y = f(x)$ has a local maximum at (-1,16) and a local minimum at (3, -16), then the graph with equation $g(x) = f(x-2) + 3$ has:

A. A local maximum at (-3,19) and a local minimum at (1,-13)

B. A local maximum at (-3,19) and a local minimum at (1,-19)

C. A local maximum at (2,19) and a local minimum at (2,-13)

D. A local maximum at (1,19) and a local minimum at (5,-19)

E. A local maximum at (1,19) and a local minimum at (5,-13)

Question 24

Given $g(x) = f(\log_e(x^2))$ then $g'(x) =$

A. $2f'(\log_e(x^2))$

B. $2xf'(\log_e(x^2))$

C. $\frac{2}{x}f'(\log_e(x^2))$

D. $\frac{2x}{\log_e(2x)}$

E. $\frac{2}{x}$

Question 25

For the graph with the rule: $y = \sqrt[3]{f(x)+1}$, $\frac{dy}{dx}$ is equal to:

A. $\sqrt[3]{f'(x)+1}$

B. $f'(x)\sqrt[3]{f(x)+1}$

C. $f'(x)\frac{2}{\sqrt{f(x)+1}}$

D. $f'(x)\frac{1}{3(\sqrt[3]{f(x)+1})^2}$

E. $3f'(x)\sqrt{f(x)+1}$

Question 26

If the hybrid function $f(x)$ is defined by:

$$f(x) = \begin{cases} 2x, & x \leq 1 \\ \frac{1}{2}x^2 + bx + c, & x > 1 \end{cases}$$

A. $b = 1$ and $c = -\frac{1}{2}$

B. $b = -1$ and $c = -\frac{5}{2}$

C. $b = 2$ and $c = -\frac{5}{2}$

D. $b = -1$ and $c = \frac{5}{2}$

E. $b = 1$ and $c = \frac{1}{2}$

Question 27

Let $h'(x) = \cos(g(x))$. If $g'(\frac{\pi}{6}) = \frac{1}{3}, g(\frac{\pi}{6}) = \frac{\pi}{2}$ and $h(\frac{\pi}{6}) = 2$, find the rule of $h(x)$.

A. $h(x) = \frac{1}{g'(x)} \sin(g(x)) - 1$

B. $h(x) = -\frac{1}{g'(x)} \sin(g(x)) - \frac{3}{2}$

C. $h(x) = -g'(x) \sin(g(x)) + \frac{3}{2}$

D. $h(x) = \frac{1}{g(x)} \sin(g(x)) - 1$

E. $h(x) = -\frac{\pi}{2g'(x)} \sin(g(x)) + 2$

Question 28

Given c is a constant value, $P'(x) = f'(x)\cos(f(x))$, then $P(x) =$

A. $-\cos(f(x)) + c$

B. $\cos(f(x)) + c$

C. $\sin(f(x)) + c$

D. $-\sin(f(x)) + c$

E. $f(x) \times \cos(f(x)) + c$

Question 29

Consider the polynomial function that is continuous and smooth for all $x \in \mathbb{R}$ and has the following features:
- $f'(x) = 0, x \in \{2, 7, 10\}$
- $f'(x) < 0, x \in (-\infty, 2) \cup (2, 7) \cup (10, \infty)$
- $f'(x) > 0, x \in (7, 10)$

Which of the following statement is true about $f(x)$.

A. $f(x)$ has a stationary point of inflection at $x = 2$, a local minimum at $x = 7$ and a local maximum at $x = 10$.

B. $f(x)$ has a stationary point of inflection at $x = 2$, a local maximum at $x = 7$ and a local minimum at $x = 10$.

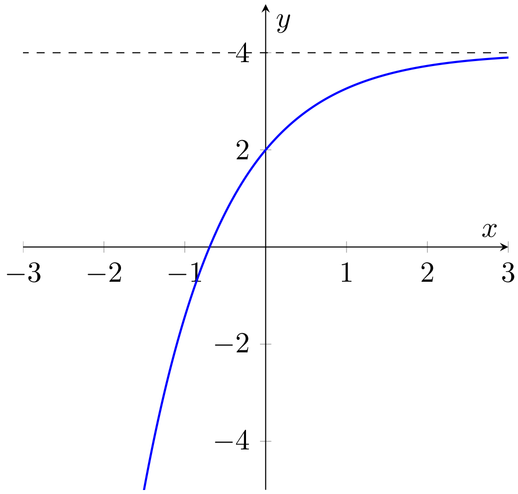
C. $f(x)$ has x-intercepts at $x = 2, x = 7$ and $x = 10$.

D. $f(x)$ has x-intercepts at $x = 2$ and $x = 7$ a local maximum between $x = 7$ and $x = 10$.

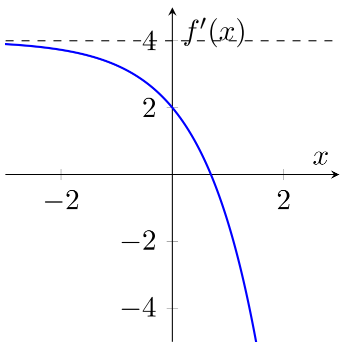
E. $f(x)$ has local maximums at $x = 2$ and $x = 10$ and a local minimum at $x = 7$.

Question 30

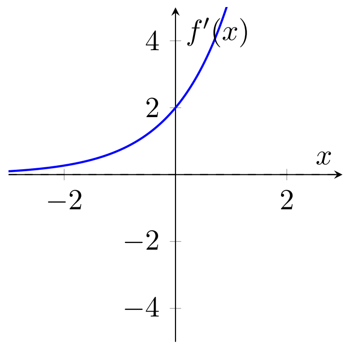
Part of the graph of the function with equation $y = f(x)$ is shown right. Which of the following is most likely to be the corresponding part of the graph of the function with equation $f'(x)$.



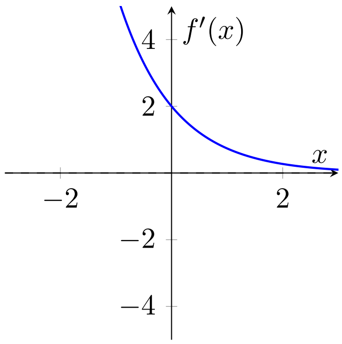
A.



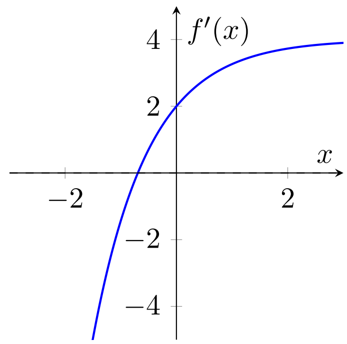
B.



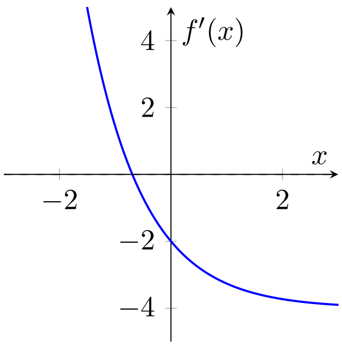
C.



D.

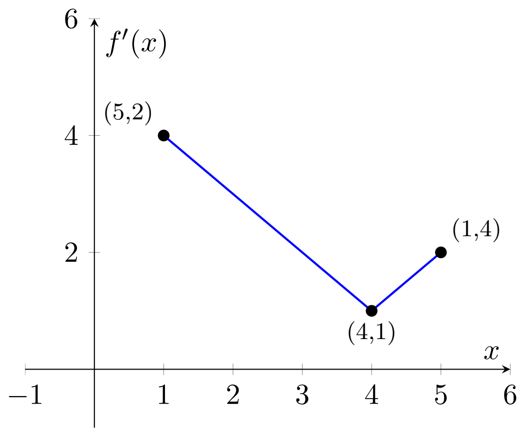


E.



Question 31

The graph of the function h is shown below.



The average value of h is:

A. $\frac{9}{16}$

B. $\frac{5}{2}$

C. $\frac{9}{2}$

D. $\frac{7}{4}$

E. $\frac{9}{4}$

Question 32

If the cubic function $f(x) = (x-2)(x^2-4x+c)$ has 3 distinct x intercepts, then:

A. $c > 2$

B. $c \geq 2$

C. $2 \leq c \leq 4$

D. $c < 4$

E. $4 < c$

Question 33

A cubic function has the rule $y = f(x)$. The graph of the derivative function $y = f'(x)$ crosses the x-axis at (2,0) and (-3,0). The maximum value of the derivative function is 10. The value of x for which the graph of $y = f(x)$ has a local maximum is:

A. $x = -2$

B. $x = 2$

C. $x = -3$

D. $x = 3$

E. $x = 10$

Question 34

At $x = 1$, the hybrid function $f(x) = \begin{cases} 3x - 3, & x \geq 1 \\ x^2 + 2x - 3, & x < 1 \end{cases}$

A. Is continuous and differentiable

B. Is continuous, but not differentiable

C. Is not continuous, but differentiable

D. Is not continuous and not differentiable

E. Has a stationary point (1,0)

Question 35

If $u(x) > 0$ for $x \in [-2, 1]$ and $\int_{-2}^{1} u(x)\, dx = 8$, then $\int_{3}^{0} \left(\frac{u(x-2)}{2} + 2 \right)\, dx$ is equal to:

A. -10

B. -7

C. -6

D. -2

E. 2

Question 36

Given $f(x)$ is a linear function, the anti-derivative of $\frac{2}{(3-f(x))}$ is:

A. $-\frac{2}{f'(x)}\log_e|3-f(x)|+c$

B. $-2\log_e|3-f(x)|+c$

C. $-2f'(x)\log_e|3-f(x)|+c$

D. $\frac{2}{f(x)(3-f(x))^2}+c$

E. $\frac{2}{f'(x)(3-f(x))^2}+c$

Extended Response Questions

Question 1

A new theme park is developing a rollercoaster ride, where the leading engineer, Victor is developing the path that the rollercoaster will travel. While brainstorming, Peter took down the following notes:
- The rollercoaster will start and finish on the ground
- The rollercoaster ride will end at a different position to where it starts. It will deploy its passengers and then make its way back to the start of the ride again.

Victor has suggested a part of the rollercoaster's path should follow the equation:

$$h(x) = 6\sin\left(\tfrac{\pi}{2}(x-2)\right) + 15, \quad 6 \leq x \leq 28$$

Where h m is the height of the rollercoaster at any given time t seconds above the ground.

a) State the maximum height of the rollercoaster above the ground.

b) State the minimum height of the rollercoaster above ground level for $x \in [6, 28]$

c) Find x, where the height of the rollercoaster first reaches 17m above the ground level. Give your answers correct to three decimal places.

Victor has established that the incline path of the rollercoaster can be described by the general equation $f(x) = ax^3 + bx^2 + cx + d, \quad 0 \leq x < 6$.
Where f m is the height of the rollercoaster at given distance x m from the starting point and a, b, c and d are constants.

d) **i.** Find an expression, in terms of x, for the rate of change of h with respect to distance. Hence, find the rate of change of the height of the rollercoaster above the ground level when $x = 6$.

ii. Considering that the rollercoaster must join smoothly at the point $x = 6$, find an equation, in terms of a, b and c equating the rate of change of f and h at this point.

iii. Considering that the rollercoaster begins at ground level with zero gradient, find the exact values of a, b, c and d.

Question 2

A particle has acceleration given by $a(t) = 2t - 4$, where a is the acceleration in metres per second squared and time t is in seconds.

a) What is the particle's initial acceleration ?

b) The particle's initial velocity is -10ms^{-1} and its initial position is 0m.

i. Find the particle's velocity, $v(t)$, at time t seconds.

ii. Find the particle's displacement, $x(t)$, at time t seconds.

c) Find the particle's average velocity over the first 10 seconds.

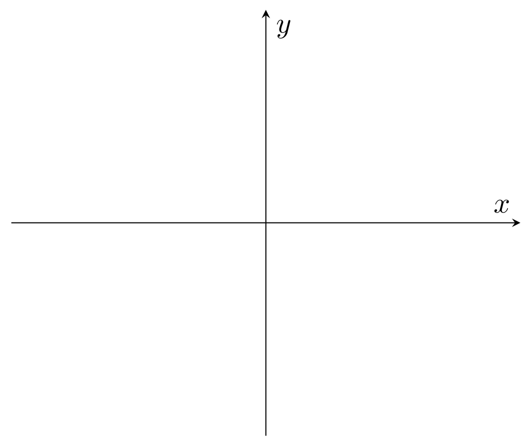
d) Within the first 10 seconds, find the particle's acceleration and displacement when its velocity is 0 ms^{-1}.

e) Find the distance travelled over the first $\sqrt{14} + 2$ seconds.

Question 3

Let $f : [1, 3] \to \mathbb{R}, f(x) = \sqrt{x-1}$

a) Sketch the graph of $fx)$ on the axis provided

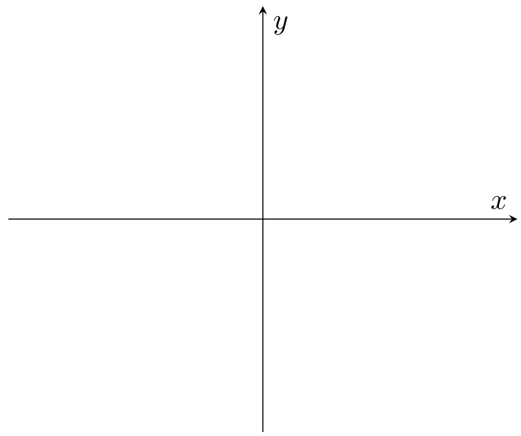


b) Use the right endpoint method with rectangles of 4 equal widths to approximate the area between $f(x)$ and the x-axis correct to 4 decimal places. Sketch the right endpoint rectangle onto the graph drawn in part a. Label the top left corner of each rectangle formed with its coordinates.

c) Use calculus to find $\int_1^3 \sqrt{x-1}\, dx$, giving your answer as an exact value.

d) Show that the rule for the inverse function $f^{-1}(x)$ is $f^{-1}(x) = x^2 + 1$. Write the domain of this function.

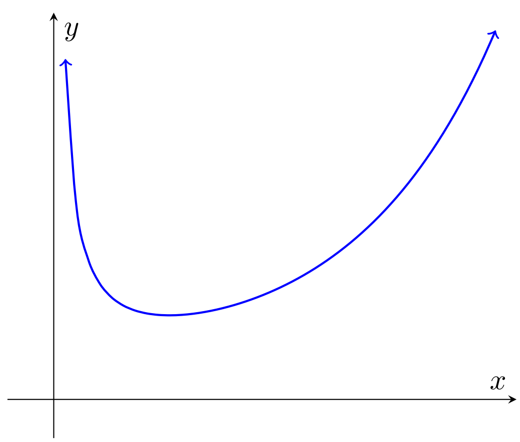
e) Sketch the graph of $f^{-1}(x)$ on the axis provided.



f) Hence, using calculus, find $\int_0^{\sqrt{2}} f^{-1}(x)\ dx$, giving your answer as an exact value.

Question 4

The graph of $y = f(x)$ where $f(x) = \dfrac{e^{4x}}{x}$ is sketched below over the domain $\mathbb{R}^+$.



a) Find the derivative of $f(x)$.

b) Hence find the exact coordinates of the minimum.

c) On the same set of axes as the diagram shown above, sketch the graph of $f'(x)$.

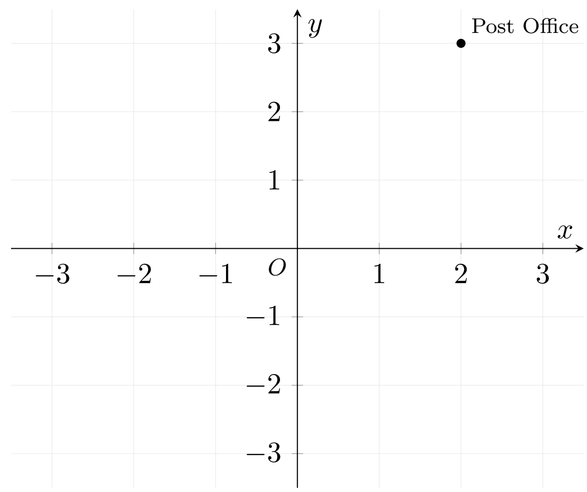
d) Write down the interval on which $f(x)$ is increasing.

e) **i.** Find the exact coordinates of the intersection of the graphs $y = f(x)$ and $y = f'(x)$.

ii. Hence find the equation of the tangent to $y = f(x)$ at the intersection point.

Question 5

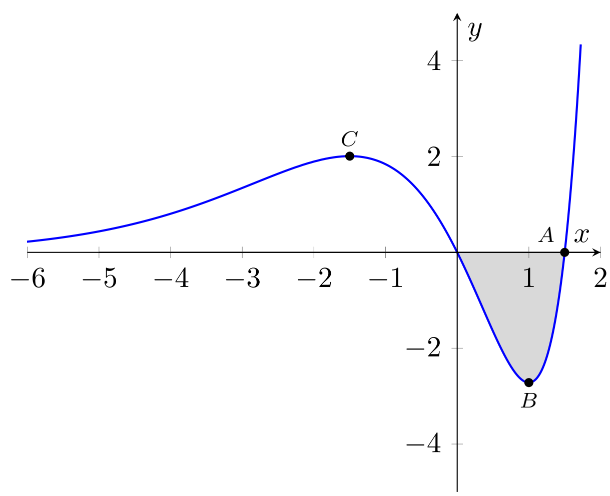
In a country town, it is decided that a new road should be built. The grid below shows the positions of the railway line and the Post Office. In each direction, 1 unit represents 1 kilometre.
It is decided that the road should follow the path whose equation is:



$$y = (2x^2 - 3x)e^{ax}, \text{ where } a > 0.$$

a) Find the value of a for which the road will pass through the Post Office. Give your answer correct to three decimal places.

In fact, they decide to build the road for which $a = 1$ as shown in the diagram below.



b) Find the x-coordinate of the point A where the road crosses the railway line.

c) Use calculus to find the coordinates of the turning point B. Give your answer correct to three decimal places.

The town council wishes to develop the shaded area bounded by the road and the railway line as a lake for water birds.

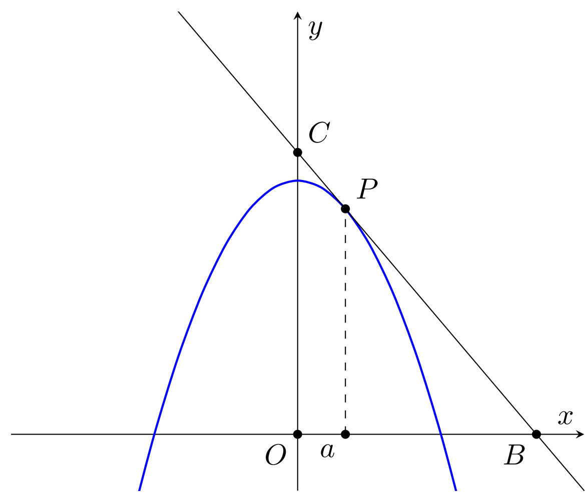
d) Find the values of m and n for which.

$$\frac{d}{dx}\{(2x^2 + mx + n)e^x\} = (2x^2 - 3x)e^x$$

Hence, find the exact area of the lake.

Question 6

The tangent T, to the curve $y = 9 - 4x^2$ crosses the x-axis at B and crosses the y-axis at C, as shown in the diagram. O is the origin.
The coordinates of B and C are $(b, 0)$ and $(0, c)$ respectively, where b and c are positive real numbers.



a) Show that the equation of the tangent, T, to the curve $y = 9 - 4x^2$ at the point P, where $x = a, 0 < a < \frac{3}{2}$

b) Hence, show that $c = 4a^2 + 9$

c) Show that $b = \dfrac{4a^2 + 9}{8a}$

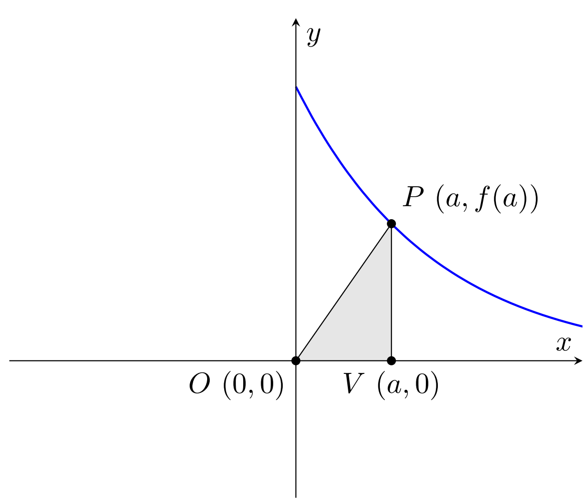
d) Express the area, A of the triangle BOC in terms of a.

e) Find the exact value of a for which the area A of the triangle is a minimum.

f) Verify that it is a minimum and find the exact minimum area of the triangle BOC.

Question 7

The function $f : [0, 4] \to \mathbb{R}, f(x) = 2^{2-x}$ is sketched to the right. A point P is placed on the function at $(a, f(a))$ and a right triangle is formed by the vertices; O (at the origin), the point P and the point V (which is directly below P on the x-axis).



a) Show that the area A of the triangle is given by $A = \dfrac{a \times 2^{2-x}}{2}$

b) Find $\dfrac{dA}{da}$.

c) Find the exact value of a for which the right angle triangle has the maximum possible area.

d) Write a definite integral expression that gives the total area under the curve $y = f(x)$ from $x = a$ to $x = 4$.

e) Find to one decimal place, the value for a which will make the area indicated in part d. equal to the area of the triangle.

f) Let the acute angle POV be θ. Write the rule for the magnitude of θ in terms of a.

Question 8

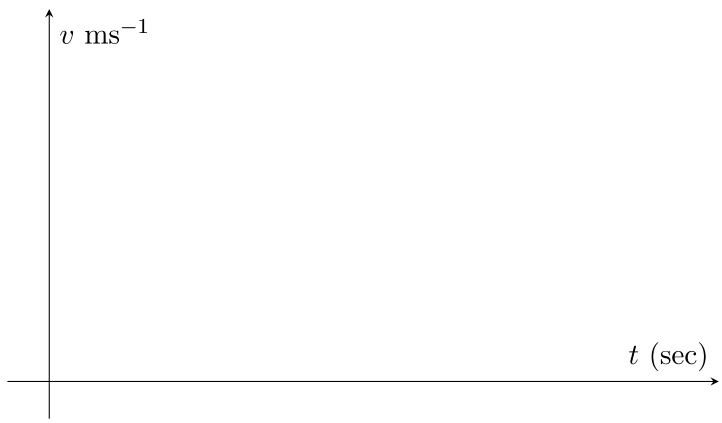
The acceleration $a(t)$ ms^{-2}, of a particle travelling in a straight line is given by the rule:

$$a(t) = 6 - \frac{2}{(t+1)^2}, \quad t \geq 0$$

where t is the time in seconds from the start.
Initially the particle is at rest.

a) Find the expression for the velocity, $v(t)$ ms^{-1}, of the particle at time, t seconds.

b) Sketch the graph of $v(t)$ against t for the first 5 seconds, showing the exact coordinates of the endpoints.



c) Find the average velocity in the first 3 seconds. State your answer correct to 2 decimal places.

d) Find the average rate of change of the velocity in the first 3 seconds.

e) **i.** Explain why the distance the particle travels in the first 4 seconds and the displacement of the particle in the first 4 seconds are the same.

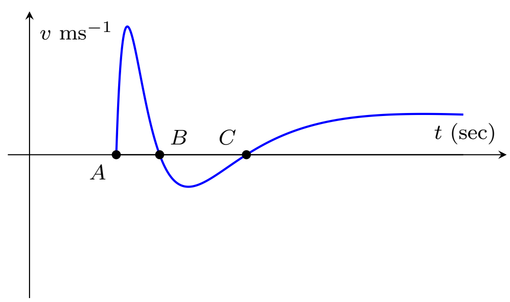
ii. Find the distance the particle travels in the first 4 seconds, correct to 2 decimal places.

Question 9

A country has a railway running along the x-axis (for $x \geq 1$) and a freeway winding across the railway.

This freeway has equation $y = f(x) = \dfrac{(x-1)(x-3)(x-5)}{x^5}$ (for $x \geq 1$).

This is shown below



At the points where the freeway crosses the railway, towns have been built and denoted A, B and C.

a) **i.** Write down $(x-1)(x-3)(x-5)$ in expanded form.

ii. Hence, showing all working calculate $\displaystyle\int \frac{(x-1)(x-3)(x-5)}{x^5}dx$.

b) All the land between the railway and the freeway was once owned by King Khong. He passed away and bequeathed the land to form parkland.

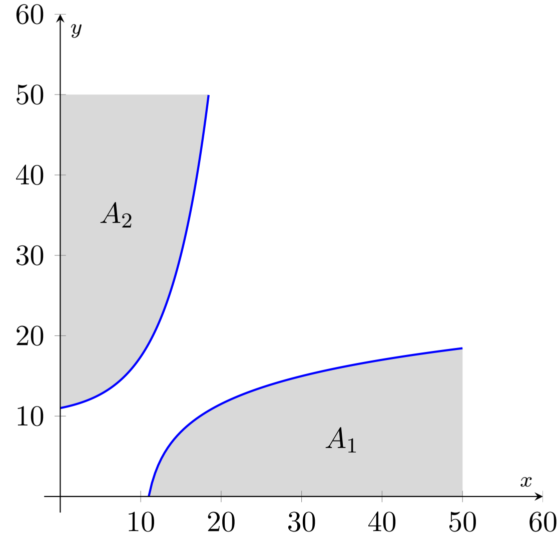
i. Write down the integral (or integrals) which when evaluated will find the area of the land contained between the railway and freeway between A and C in units squared.

ii. Hence, by evaluating the integral (or integrals), find the exact area of the land contained between the railway and the freeway between A and C. Give your answer in km^2, with one unit equal to one kilometre.

c) How much land did King Khong bequeath in total (in km^2 accurate to 4 decimal places).

Question 10

The Limmer is a spy. She is attempting to flee enemy territory and begins her escape at point O (0,0) shown in the diagram below.
The x-axis runs in the east-west direction. Part of the ground she must run through is floodlit and the floodlit areas are shaded in the diagram above. All measurements are given in metres.
The floodlit area A_1 is enclosed by the x-axis, the line $x = 50$ and the function.
$f : [a, 50] \to \mathbb{R}, f(x) = 5\log_e(x - 10)$



a) Show that $a = 11$

The floodlit area A_2 is enclosed by the y-axis, the line $y = 50$ and the graph of the function f^{-1}, the inverse of f.

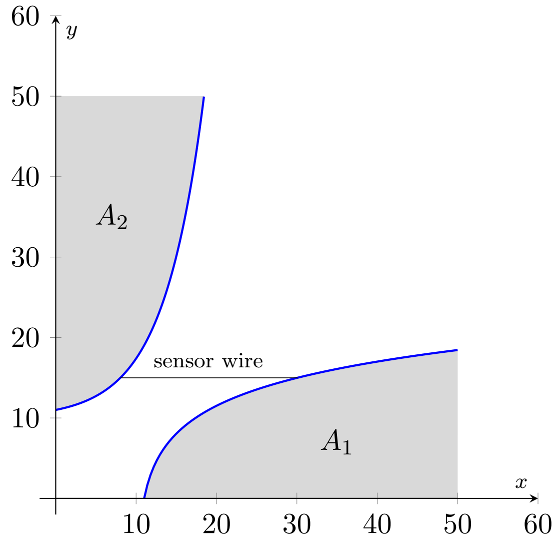
b) Find the rule of f^{-1} and state its domain and range.

c) **i.** Find the exact area bounded by $f^{-1}(x)$, the x-axis and the lines $x = 0$ and $x = 5\log_e 40$.

ii. Use the answer above to find A_2.

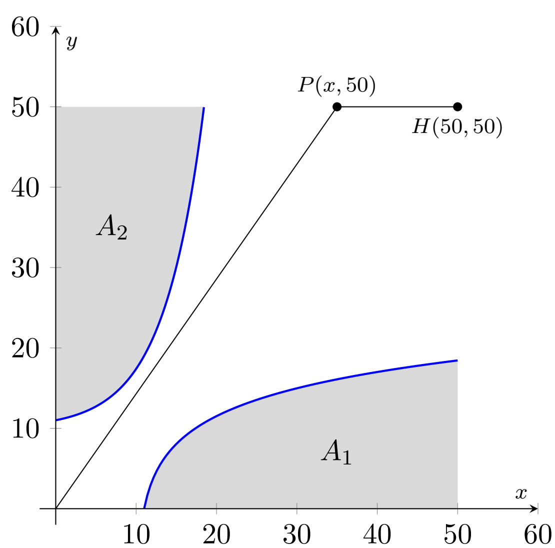
iii. Find the total area of the floodlit ground to the nearest square metre.

The Limmer moves in a straight line from $O(0,0)$ and without knowing, passes over a sensor wire that runs in an east-west direction along the line $y = 15$ in the area that is not floodlit.
The point where she passes over the sensor is given by $(b, 15)$.



e) Given that Limmers can move through the floodlit area if necessary, find the possible values of b correct to 2 decimal places where appropriate.

Limmers continues to move in a straight line from her starting point $O(0,0)$ until she is at a point $P(x, 50)$. From this point she moves due east to a waiting helicopter at $H(50, 50)$.



The time T, in seconds, taken by Limmers to move from O to H via P is given by:

$$T = \sqrt{x^2 + 2500} + \frac{50 - x}{2}, x \in [0, 50]$$

f) **i.** Use calculus to find the value of x correct to two decimal places for which Limmers reaches the helicopter in minimum time.

ii. Hence, find the minimum time T correct to 2 decimal places.

Question 11

a) Given the equation $f(x) = \sin\left(\frac{\pi x}{2}\right) + \cos\left(\frac{\pi x}{2}\right),\ 0 \leq x \leq 2\pi$, use algebra to show that the first point of intersection with positive x-axis occurs at $x = \frac{3}{2}$.

b) Find $f'(x)$.

c) Hence, use calculus to show that the first stationary point which occurs after $x = 0$ will occur at $x = \frac{1}{2}$ and determine the nature of this stationary point.

d) Use calculus to determine the area enclosed between $f(x)$ and the x-axis between $x = \frac{1}{2}$ and $x = \frac{3}{2}$. Give your answer in exact form.

Question 12

A half cylinder water trough is to be made from a sheet of metal (excluding lid).

a) Write down a formula for the column (V) of the term in terms of l and w. (Hint: Volume of a cylinder, $V = \pi r^2 h$)

b) If the maximum volume (capacity) of the trough is to be 20 litres (20000 cm^3), show that $l = \dfrac{160000}{\pi w^2}$.

c) Show that the formula for the area (A cm^2) of sheet metal to be used is $A = \dfrac{\pi w}{4}(w + 2l)$.

d) Obtain formula for area (A) in terms of w.

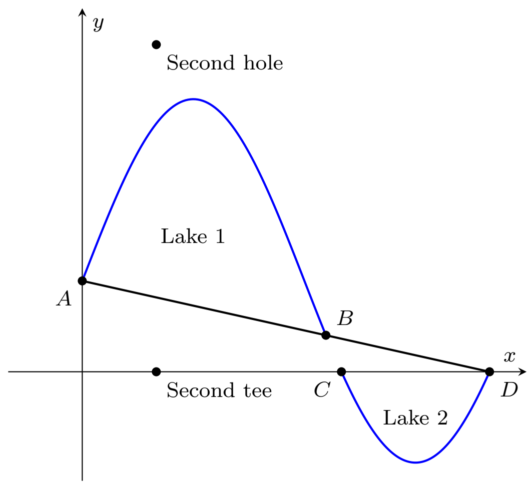
e) Use calculus to find an equation in w which applies for minimum area.

f) Solve this equation to evaluate the exact value of w for which the area is a minimum.

Question 13

Steve is playing golf at a gold course. The diagram below shows the layout of the course in the vicinity of the second hole. A set of Cartesian axes has been superimposed on the course layout. Distances along the x and y axes are measured in metres.

The curved edges of both the lakes can be modelled by the equation: $y = 100\sin\left(\frac{\pi x}{150}\right) + 50$. There is a straight road between the points A and D. This road forms a straight edge for Lake 1 while the x-axis forms the straight edge of Lake 2.



a) Show algebraically that the coordinates of C and D are $(175, 0)$ and $(275, 0)$ respectively.

b) Write down the definite integral that represents the area of Lake 2

c) Use your calculator to determine the area of Lake 2.

d) What is the equation of the line joining the points A and D.

e) Use your calculator to find the coordinates of B (correct to one decimal place)

f) Write down a definite integral that represents the surface area of Lake 1. Use this to determine the area of Lake 1 to two decimal places.

g) The second tee is located at $(50, 0)$, and second hole at $(50, 180)$. Steve hits a perfectly straight drive which results in a hole in one. What is the horizontal distance that the ball travels over the water?

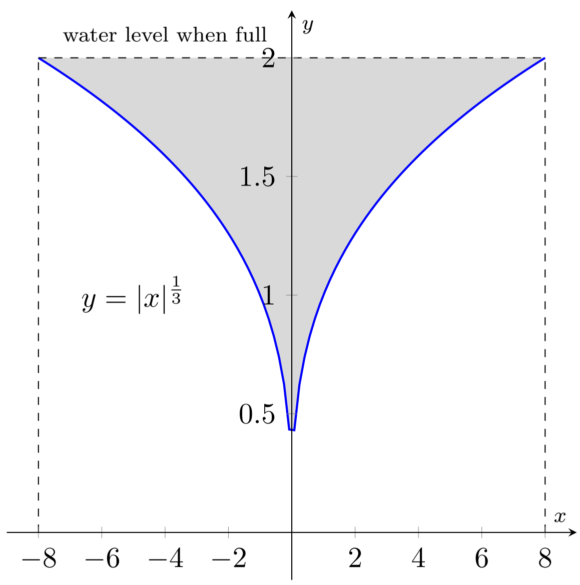
h) If the second tee is located on the x-axis at $(a, 0)$, where $0 < a < 150$ and the second hole is at $(a, 180)$, find an expression in terms of a for the horizontal distance of the straight drive like the one in part g. will travel over the water.

i. Using calculus, find the value of a that will maximise the distance such a drive travels over the water.

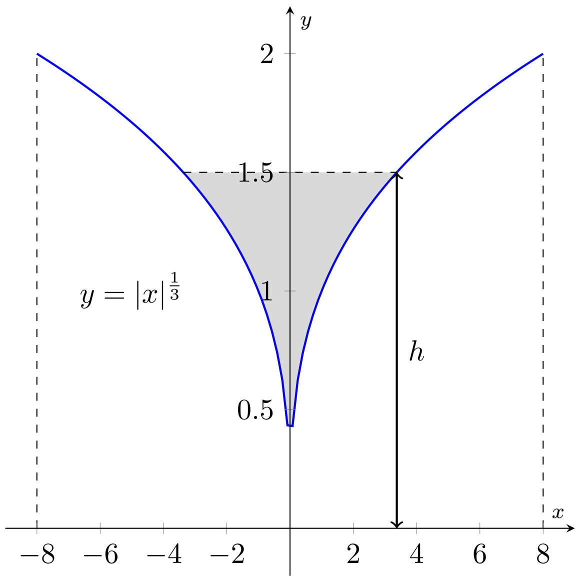
Question 14

A fish tank is 1m long and has a uniform cross-section (which is sketched) bounded by $y = 2$ and $y = |x|^{\frac{1}{3}}$ for $x \in [-8, 8]$.

a) Find the volume of the fish tank in terms of h.



b) **i.** Show that if the fish tank is filled to a depth of h metres (where h ≥ 0) the volume of water V contained is given by: $V = \dfrac{1}{2}h^4$.



ii. Find the depth h of the water in the fish tank as a function of volume V.

c) Water is entering the fish tank at a rate of $9e^{-3t}$ m^3/min.

i. Write down the rule $\dfrac{dV}{dt}$, the rate of change of volume of water in the tank.

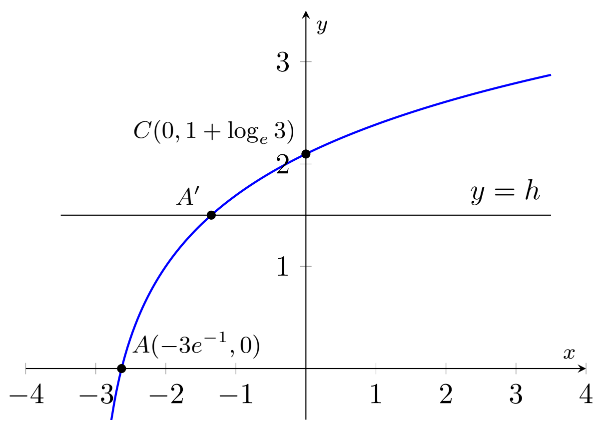
ii. If the fish tank is initially empty at time $t = 0$, find the volume $V(t)$ at any time t.

d) **i.** Show that $h(t) = (6(1 - e^{-3t}))^{\frac{1}{4}}$.

ii. What is the long term depth of the water.

Question 15

Consider the sketch below of the line with rule $y = h$ and the curve with rule $y = 1 + \log_e(x + 3)$. The point A with coordinates $(-3 + e^{-1}, 0)$ is the x-intercept of this curve, the point C with coordinates $(0, 1 + \log_e 3)$ is the y-intercept of the curve and the intersection point of the line $y = h$ and the curve $y = 1 + \log_e(x + 3)$ is labelled A'.



a) Find the area between the x-axis, the y-axis and the curve $y = 1 + \log_e(x + 3)$.

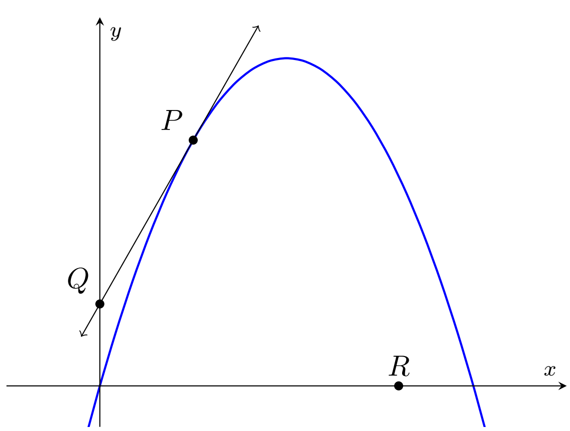
b) Find, in terms of h, the x-coordinate of the points A'.

c) **i.** Find the values of h (to four decimal places), such that the area enclosed by the graphs $y = h$, to the left of the y-axis and below the curve $y = 1 + \log_e(x + 3)$ is half the area of the region found in part a.

ii. Find the average value of $y = 1 + \log_e(x + 3)$ over $[-3 + e^{-1}, 0]$.

Question 16

Consider the curve $y = x(4 - x)$ which is sketched below.



a) **i.** In the diagram, label both intercepts and the turning points with their coordinates.

ii. Find $\dfrac{dy}{dx}$.

iii. Find the equation of the normal to the curve at the point on the curve where $x = 1$.

b) Consider now the point P with coordinates $(a, a(4-a))$, where $a \in [0, 2)$.

i. Find, in terms of a, the gradient of the curve $y = x(4-x)$ at P.

ii. Find, in terms of a, the equation of the tangent line to the curve $y = x(4-x)$ at the point P.

iii. Let Q be the y-intercept of this tangent line. Find, in terms of a, the y-coordinate of Q.

c) **i.** Consider the line L with rule:

$$y = \frac{1}{2(a-2)}x - \frac{a(2a^2 - 12a + 17)}{2(a-2)}$$

Using the gradient, show that the line L is perpendicular to the tangent to the curve $y = x(4 - x)$ at the point P.

ii. Let R be the x-intercept of the line L. Find the x-coordinate of R.

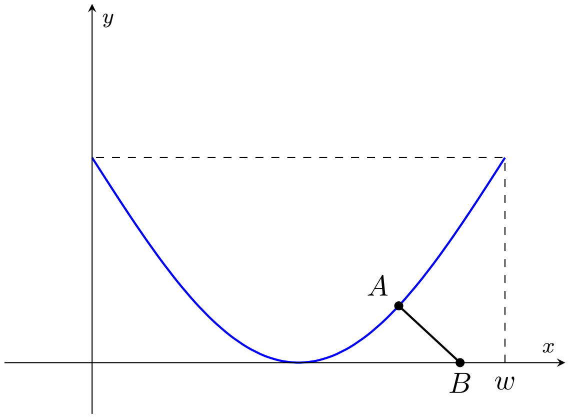
iii. Let O be the origin $(0, 0)$. Show that the area of the triangle OQR is given by:

$$A = \frac{(2a^2 - 12a + 17)a^3}{2}$$

iv. Find the exact value of a that maximises the area of the triangle OQR and write down this maximum area to 2 decimal places.

Question 17

Parko is a competitive skateboarder and, so that he can practise at home, he is building his own skateboard ramp. It is exactly 2 metres high and has the cross-section given by the curve $y = 2 - 2\sin\left(\frac{2x}{3}\right)$ as shown in the diagram.



a) Find the value of w, the width of the ramp. Give your answer to two decimal places

Although he is an expert, Parko doesn't want the ramp too steep.

b) Find $\frac{dy}{dx}$ and hence determine the maximum gradient of the ramp.

c) His coach, Jarryl suggests this is too steep. Write down the equation of an alternative ramp, in the form $y = 2 - 2\sin(bx)$, where $b \in \mathbb{R}$, which will be less steep than the one Parko plans.

Unfortunately, Parko does not think the ramp is too steep and proceeds with his original design.

d) He intends to paint the ends of the ramp to protect the timber from weather. Write down the integral which can be used to find how many square metres needs to be painted.

e) Hence, find the area painted correct to two decimal places.

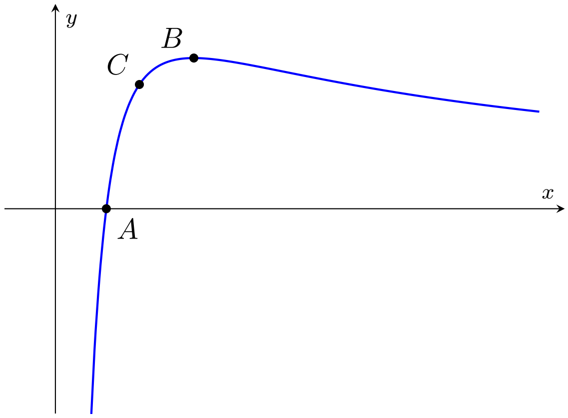
There is a supporting beam on the structure as shown. A is a point exactly one metre above ground level and B is a point on the x-axis such that AB is a normal to the curve at A.

f) Show that the x-coordinate of A is $\dfrac{5\pi}{4}$.

g) Find the exact gradient of the normal to the curve at A.

Question 18

The graph below is typical of the family of graphs of function $f : \mathbb{R}^+ \to \mathbb{R}, f(x) = \dfrac{k \log_e(x)}{x}$.



A is the x-intercept, B is the stationary point and C is the point where $x = \sqrt{e}$.

a) **i.** Find the coordinates of A.

ii. Find $f'(x)$ in terms of k.

iii. Hence find the x-coordinate of B.

b) **i.** Find the equation of the tangent at C for any k.

ii. What is the x-intercept of the tangent line?

c) **i.** Let $g(x) = (\log_e(x))^2$. Find $g'(x)$.

ii. Hence, find the anti-derivative of $\dfrac{\log_e(x)}{x}$.

iii. Write down the definite integral which gives the area of the region bounded by the graphs of $y = \dfrac{\log_e(x)}{x}$, $y = 0$ and $x = e^2$.

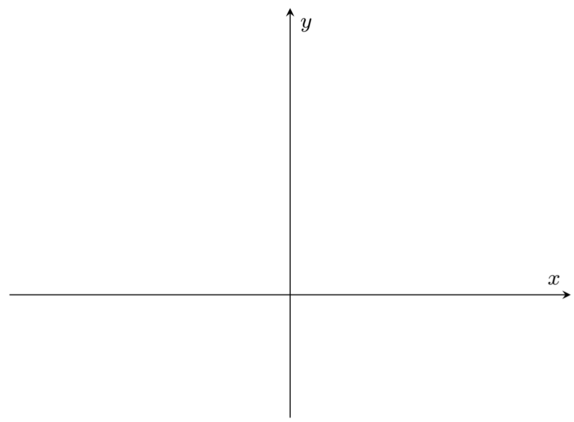
iv. Hence, calculate the area of the region bounded by the graphs of $y = \dfrac{\log_e(x)}{x}$, $y = 0$ and $x = e^2$.

v. Find the exact value(s) of m, such that the area of the region bounded by the graphs of $y = \dfrac{\log_e(x)}{x}$, $y = 0$ and $x = m$ is 1 square unit.

Question 19

A canoe has a uniform cross-section modelled by the rule: $h(x) = 1 - \cos(\pi x), -\dfrac{1}{2} \leq x \leq \dfrac{1}{2}$. The hull is 2m long and assumed to have no thickness.

a) **i.** Sketch the graph of the cross-section of the hull, labelling endpoints and turning points.



ii. The waterline is d metres above the origin. Find the x-coordinates of the points where the waterline and the curve $y = h(x)$ intersect and hence show that the total width of the hull at the waterline is $\dfrac{2}{\pi}\cos^{-1}(1-d)$ metres.

If the height of the waterline above the bottom of the boat is d, then the volume of water displaced is:

$$2 \times \int_a^b I(x)\ dx$$

iii. Give the appropriate values of the terminals a and b and the integrand $I(x)$ in the definite integral above.

If X is the volume of the water displaced by the hull in m^3 then:

$$X = \frac{2}{\pi}(2(1-d)\sin^{-1}(1-d) + 2\sqrt{d(2-d)} - \pi(1-d))$$

where d is the height of the waterline above the bottom of the hull.

iv. The volume of the water displaced by the hull is $0.04\ \mathrm{m}^3$. Find the height of the waterline above the bottom of the boat in metres accurate to 4 decimal places.

For stability reasons the hull is only safe if the waterline is at least 30cm above the floor of the canoe.

v. What is the least total volume of water displaced by the hull in order for the hull to be stable ? Give your answer in m^3 accurate to 4 decimal places.

vi. What is the greatest total volume of the water displaced by the hull before it sinks ?

b) The hull springs a leak at 1.00pm and initially the water is flowing in at the rate of $\frac{1}{10}$ m^3 per minute. The leak slowly gets larger. Prior to the 1.00pm, the hull contains no water. The rule for the rate of the leak is:

$$\frac{dV}{dt} = \frac{1}{5} - \frac{1}{10(1+t)}$$

i. Find the rule $V(t)$ for the volume of water m^3 that has leaked into the hull from 1.00pm, in terms of t, the elapsed time in minutes after 1.00pm.

ii. Find the time it takes in minutes (accurate to 4 decimal places) for the leak to fill the hull (causing the hull to sink).