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Specialist Maths Units 3/4

Calculus

Practice Questions

- Differentiation and rational functions
- Techniques of Integration
- Applications of Integration

Short Answer Questions

Question 1

Consider the family of curves defined by the relation $3x^3 - y^2 + kx + 5y - 2xy = 4, k \in \mathbb{R}$

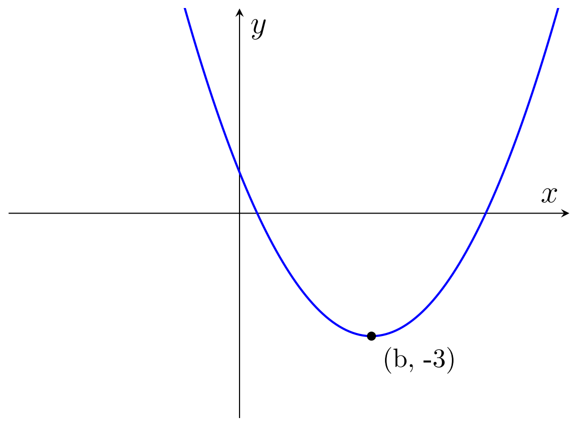
a) Verify that every curve in the family passes through the point (0,4), and find the other point of intersection with the y-axis.

b) Find an expression for $\dfrac{dy}{dx}$ in terms of x, y and k.

c) Hence, evaluate the gradient of the curve at the point (1,1).

Question 2

The sketch below shows the graph of a function $y = f(x)$. On the same axes, sketch the graph of $y = \dfrac{1}{f(x)}$ indicating clearly any asymptotes, turning points and intersections with the graph of $y = f(x)$, which should be labelled with their coordinates.



Question 3

a) Using implicit differentiation, find $\dfrac{dy}{dx}$ in terms of x and y in the following relation.

$$x^2y + \cos(y) = e^{2x} - 1$$

b) Hence, find the exact value of the gradient of this relation at the point $\left(0, \dfrac{\pi}{2}\right)$.

Question 4

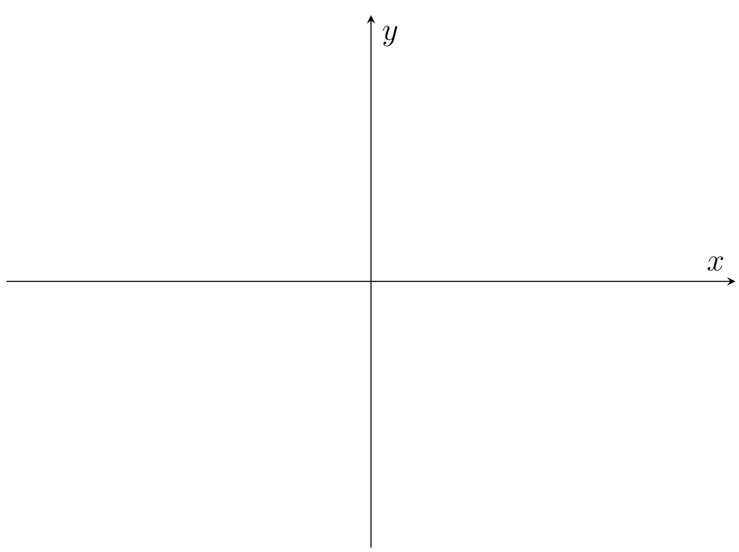
Differentiate $y = (\sin^{-1}(x))^2$.

Question 5

Consider $y = \dfrac{-3x+4}{(x-1)^2}$

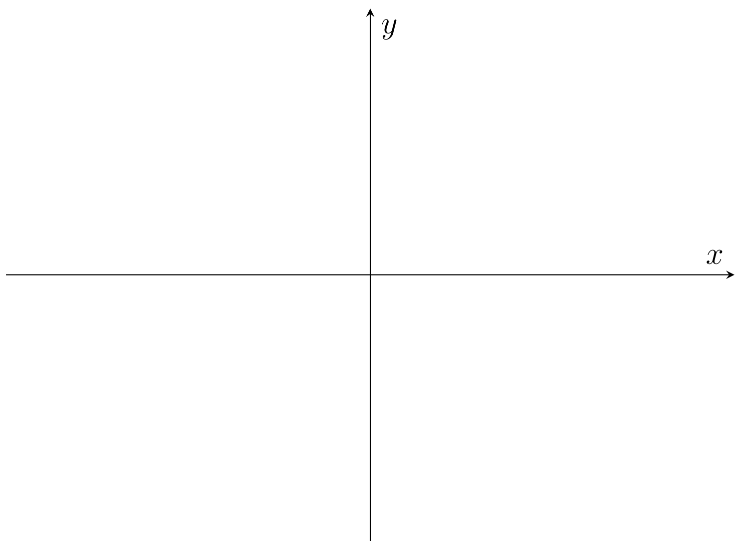
a) Express $\dfrac{-3x+4}{(x-1)^2}$ in partial fraction form. You must show your working out.

b) Graph this relation on the axis below. Clearly show all intercepts and turning points in exact coordinate form and label all asymptotes.



Question 6

Sketch the graph of $f(x) = \dfrac{3}{x^2 - 2x - 8}$ on the axes below. Give the exact coordinates of any turning points, intercepts and state the equations of all straight line asymptotes.



Question 7

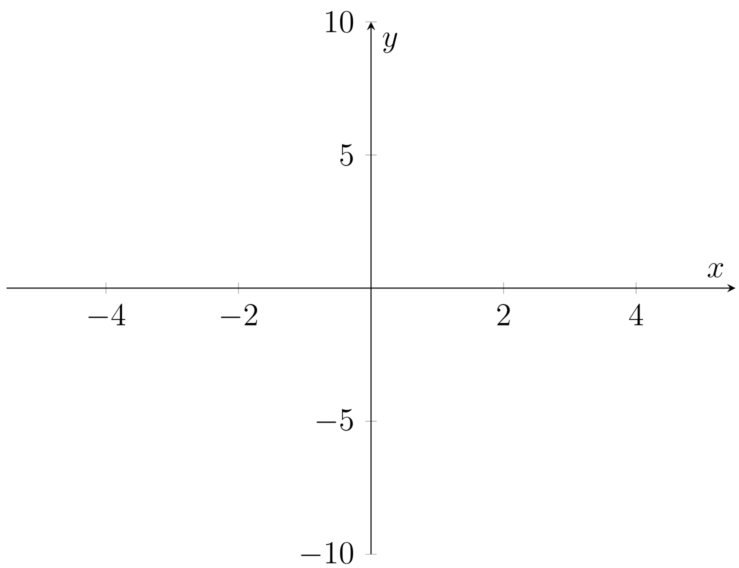
Find the value of n so that the graph $f(x) = \dfrac{1}{2x^2 - 10x + n}$ has exactly one vertical asymptote.

Question 8

Consider the function with the rule $f(x) = \dfrac{2x^3+1}{x^2}$.

a) Use calculus to determine the coordinates of any turning points on this curve.

b) Hence, sketch the graph of $f(x) = \dfrac{2x^3+1}{x^2}$ showing clearly any axial intercepts and asymptotes.

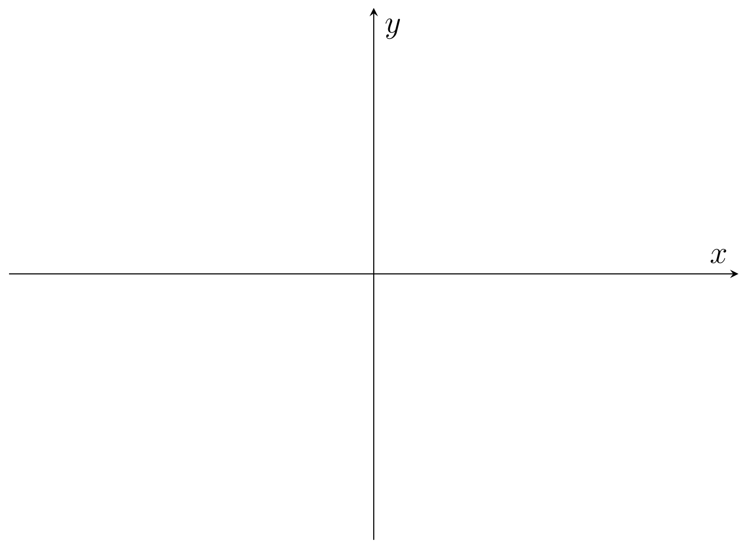


Question 9

Find $\dfrac{d^2y}{dx^2}$ if $y = \sin(e^x)$. Hence, determine whether the concavity is directed upwards or downwards when $x = 0$.

Question 10

Sketch the graph of $y = \dfrac{1}{x^2} + x - 2$, showing all intercepts and coordinates correct to 2 decimal places. Label all asymptotes.



Question 11

The curve C us defined by $x^2 - xy + y^2 = k$, where k is a real constant. It is known that C passes through the point $P = (-1, y)$.

a) At point P, find an expression for y in terms of k.

b) Find the coordinates of P for $k = 3$.

c) Find the equation of the normal line to the curve C at point P where $y > 0$.

Question 12

Find the exact value of k if

$$\int_{\frac{1}{4}}^{\frac{1}{2}} \frac{1}{\sqrt{1-4x^2}}\,dx = k\pi$$

Question 13

Given that

$$\frac{10x}{(x^2+1)(3x+1)} = \frac{x+A}{x^2+1} + \frac{B}{3x+1}$$

Find the value of A and B.

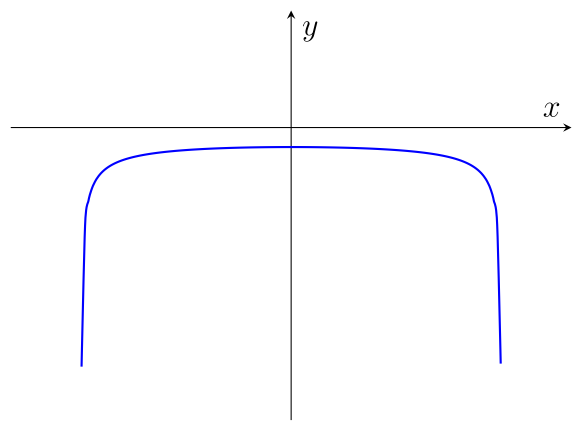
Question 14

If $3\int_0^m \frac{1}{\pi\sqrt{1-9x^2}} = \frac{1}{4}$, where $m > 0$, find the exact values of m.

Question 15

Part of the curve $f(x) = \frac{-1}{\sqrt{9-x^2}}$ is sketched on the right.

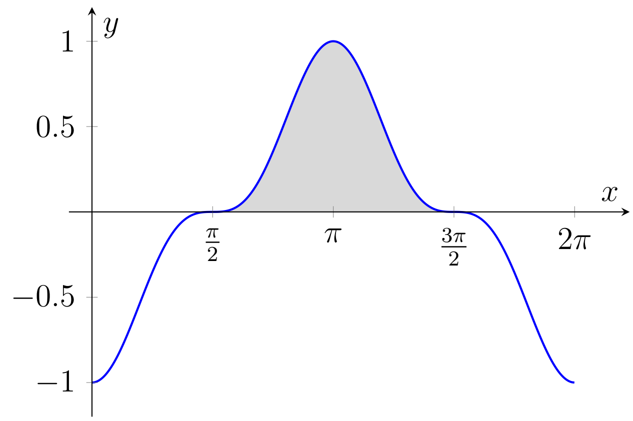
a) Using integration, find in exact form, the area of the region bounded by $y = f(x)$, the x-axis and the lines $x = \pm\frac{3\sqrt{3}}{2}$.



b) The region described in part a. is rotated about the x-axis to form a solid of revolution. Find the exact volume of this solid, giving your answer in the form $\frac{\pi}{3}\log_e(a+\sqrt{3}b)$.

Question 16

The graph of $y = -\cos^3(x), x \in [0, 2\pi]$ is shown on the right.
Find the exact area of the region above the x-axis.



Question 17

a) Find the value of A, B and C such that $\dfrac{2x+1}{x(3-x)^2} = \dfrac{A}{x} + \dfrac{B}{3-x} + \dfrac{C}{(3-x)^2}$.

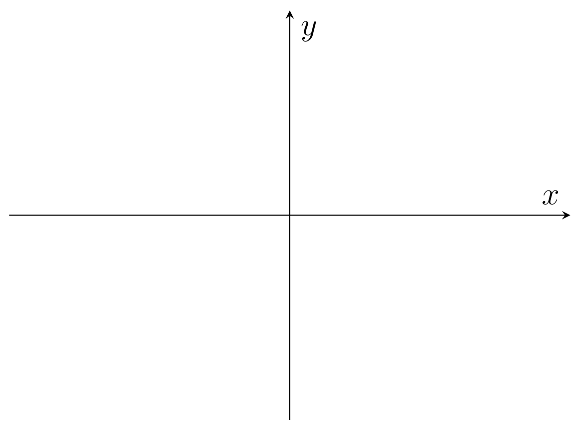
b) Hence, evaluate the integral $\displaystyle\int_{-2}^{-1} \frac{2x+1}{x(3-x)^2}\, dx$ expressing your answer in the form $m + \log_e n$, where m and n are real constants.

Question 18

Determine the exact area of the region bounded by the graph of $y = \sin^2(x)\cos^2(x)$, the x-axis and the lines $x = 0$ and $x = \dfrac{\pi}{2}$.

Question 19

a) Sketch the graph of the function with rule $f(x) = x\sqrt{9 - 2x}$, showing all intercepts and stationary points.



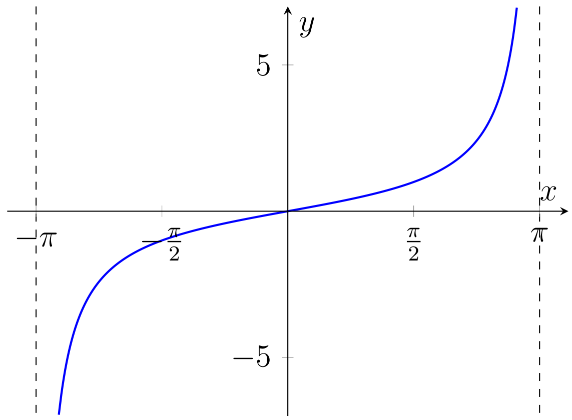
b) Find the equation of the tangent to the graph of f at the point $(4, 4)$.

c) Using calculus, find the area of the region enclosed by the graph of f, the tangent at (4,4) and the y-axis.

Question 20

The graph of the function below is $f : (-\pi, \pi) \to \mathbb{R}, f(x) = \tan\left(\frac{\pi}{2}\right)$.

a) Find the area enclosed by the graph of $y = f(x)$, the x-axis and the line $x = \frac{\pi}{3}$.

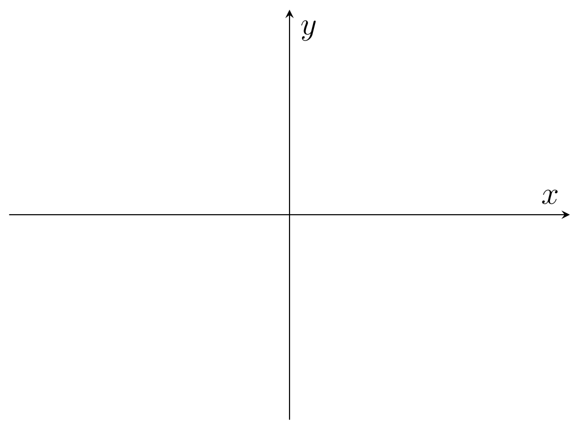


b) If the area in part a. is rotated around the x-axis to form a solid of revolution, express the volume of this solid in the form $\dfrac{\pi(a\sqrt{b} + c\pi)}{b}$.

Question 21

Consider the function with the rule $y = \sin^{-1}(x), -1 \leq x \leq 1$.

a) Sketch the graph of the function on the axes provided.



b) A solid of revolution is formed when the graph is rotated about the y-axis.

i. Write down, in terms of $\sin(y)$, an integral which can be used to find the volume of this solid.

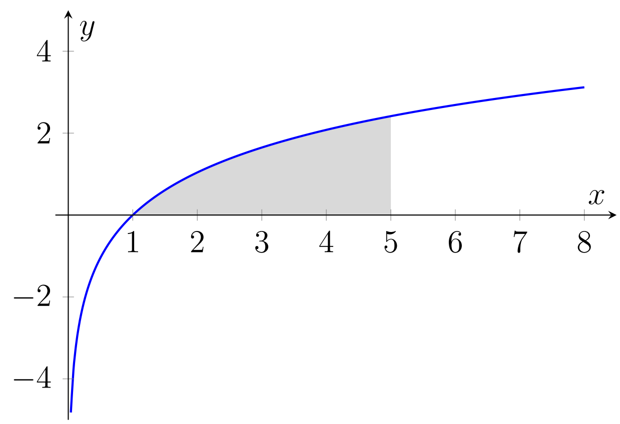
ii. Use an appropriate trigonometric identity to write your integral in a form which can be evaluated without a calculator.

iii. Evaluate your integral to find the volume of the solid, giving an exact answer.

Question 22

a) Show that $\dfrac{d}{dx}(x(\log_e(x))^2 - 2x\log_e x + 2x) = (\log_e x)^2$.

b) The graph of $y = \dfrac{3}{2}\log_e(x)$ is shown for $x > 0$. The design for a nose cone of an aeroplane is modelled by rotating the shaded region about the x-axis, between $x = 1$ and $x = 5$, where all measurements are in metres.
Use calculus to find the volume of the nose cone.



Question 23

a) Express $\dfrac{x^2}{x^2 - 4}$ in partial fraction form.

b) Use this result or other methods to find the exact volume of rotation of the function f (defined below) about the x-axis and between the lines $x = 0$ and $x = 1$.

$$f(x) = \frac{x}{\sqrt{x^2 - 4}}$$

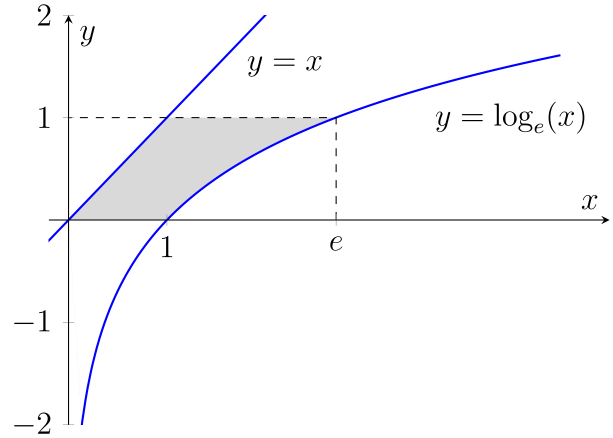
Question 24

a) Express $f(x) = \dfrac{3x^2 + x - 1}{x^2 - 1}$ in partial fraction form.

b) Hence, use calculus to find $\displaystyle\int_{-4}^{-2} \frac{3x^2 + x - 1}{x^2 - 1}\, dx$, showing all important information working and expressing your answer in the form $a + b\log_e(c)$, where a, b and $c \in \mathbb{R}$.

Question 25

Find the exact area of the shaded region.



Question 26

Determine each of the following:

a) $\displaystyle\int x\sqrt{2x+1}\,dx$

b) $\displaystyle\int e^{3x}\cos(e^{3x}-1)\,dx$

c) $\displaystyle\int_2^3 \frac{1}{x^2 - 6x + 5}\, dx$

d) $\displaystyle\int_0^1 \frac{e^x}{e^x + e^{-x}}\, dx$

Question 27

The function $y = f(x)$ is defined by $f(x) = x^2 \log_e kx$, where $k \in \mathbb{R}^+$.

a) Show that $f(x)$ has points of inflection when $x = \dfrac{1}{k} e^{-\frac{3}{2}}$.

b) Show that all inflection points $y = f(x)$ lies on the locus of a parabola if the form $y = mx^2$ where you should find the exact value of m.

Question 28

The graph of $y = \dfrac{a}{x^2 + ax + b}$ has a range of $(-\infty, 0) \cup \left[\dfrac{1}{2}, +\infty\right)$ and one of its vertical asymptotes has the equation $x = -1$. Use algebra to find the exact values of a and b.

Question 29

Compute the slope of the line that is tangent to the curve $\sin(xy) = x$ at the point $\left(\dfrac{1}{2}, \dfrac{\pi}{3}\right)$ and show that the answer is in the form

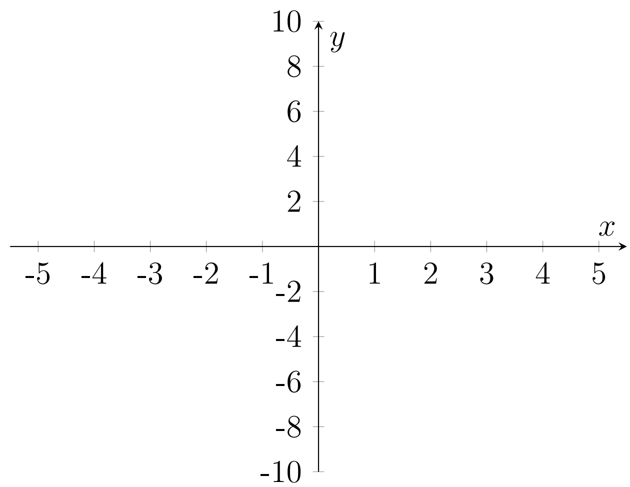
$$a\left(\frac{a\sqrt{b} - \pi}{b}\right)$$

Where a and b are constants.

Question 30

a) Find the value of k in $f(x) = \dfrac{5}{3x^2 + kx + 5}$, where $k \in \mathbb{R}^-$, such that the function has only one vertical asymptote.

b) Sketch the graph of $f(x)$ in the given axes below. Give the coordinates of any turning points and intercepts, and state the equation of all straight line asymptotes.

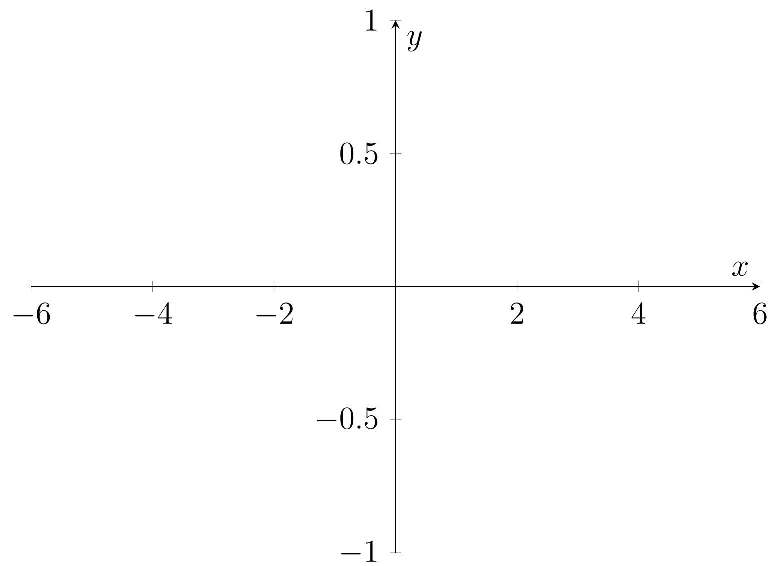


Question 31

Find an anti-derivative of $\dfrac{3x - 5}{\sqrt{25 - 9x^2}}$, for $|x| < \dfrac{3}{5}$.

Question 32

a) Use algebra and calculus to sketch the graph of $f(x) = \dfrac{x^3+2}{x} - 3$ on the set of axes below clearly labelling all axes intercepts, stationary points and asymptotes when they occur.



b) State the domain and range of the above function.

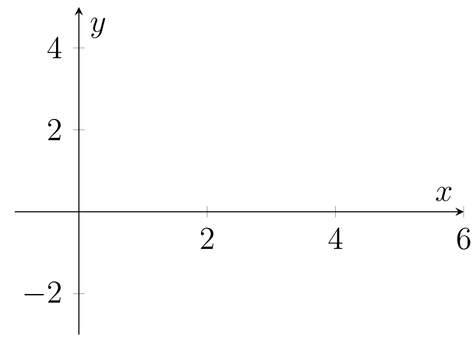
Question 33

Consider the function $f : [a, b) \cup (b, +\infty) \to \mathbb{R}$, where $f(x) = \dfrac{1}{3 - 4x + x^2}$

a) **i.** Find the smallest exact value of a such that f is a one-to-one function.

ii. Find the exact value of b.

b) For the values of a and b found prior, sketch the graph $y = f(x)$ on the axes provided.

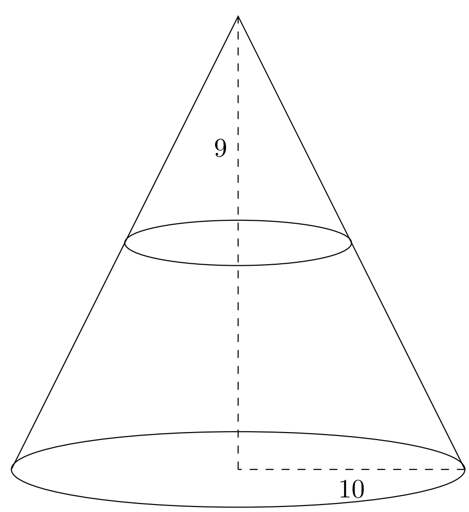


c) **i.** Use algebra to find a rule for the inverse function $f^{-1}(x)$ of $f(x)$.

ii. State the domain of $f^{-1}(x)$.

Question 34

A cone with a height of 9cm and base radius of 10cm is resting with the base on the floor as shown. Water is filling in from the apex at a rate of 9cm^3 per minute. Find the rate at which the height, h of water is increasing in the cone when h is 4cm.



Question 35

When the height of the water in a hemispherical bowl is h cm, the volume V cm^3 of water in the bowl is given by $V = \frac{1}{3}\pi h(3r - h)$, where r is the radius of the bowl.
Sarah pours water at a rate of 0.025 litres per second into a hemispherical bowl of radius 15 cm.

a) Write down $\frac{dV}{dt}$ in cm^3/s.

b) Express $\frac{dV}{dh}$ in terms of h.

c) Find the exact rate at which the water level is rising when its height is 5cm.

Question 36

a) Show that $\frac{d}{dx}\left(x\sqrt{4-x^2} + 4\sin^{-1}\left(\frac{x}{2}\right)\right) = 2\sqrt{4-x^2}$

b) Hence, give the exact value of $\int_0^1 \sqrt{4-x^2}\, dx$.

Question 37

A family of curves is defined by the relation $x^3 - xy + 4y^3 = k$, where $k \in \mathbb{R}$.
Find the value(s) of k for the curves which have:

a) a tangent parallel to the x-axis.

b) no tangent parallel to the y-axis.

c) the line $x = 3$ as a tangent.

Question 38

Let $f(x) = \dfrac{2}{\pi}\sin^{-1}\left(\dfrac{1}{2}x+1\right) - 3$.

a) State the implied domain and range of f.

b) Find $f'(x)$ giving your answer in the form $f'(x) = \dfrac{a}{\pi\sqrt{bx(x+c)}}$, where a, b and c are integers.

Question 39

Use calculus to evaluate $\displaystyle\int_1^3 \frac{x^2+1}{x^3+3x}\,dx$.

Question 40

Consider the function:

$$f : \mathbb{R}\backslash\{0\} \to \mathbb{R}, f(x) = \frac{x^3 - a}{bx^2}$$

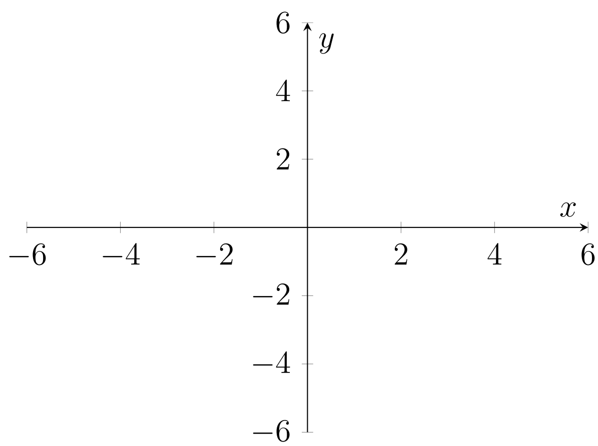
where $a, b \in \mathbb{R}$. Find the value of a and b if the graph of $y = f(x)$:

a) passes through the point $\left(-2, \frac{9}{2}\right)$ and has an asymptote with equation $y = 3x$.

b) has a minimum turning point at $(-2, -3)$.

Let $a = -1$ and $b = 2$.

c) Sketch the graph of $y = f(x)$ over its maximal domain. Label all axes intercepts and stationary points with their coordinates and all asymptotes with their equations.

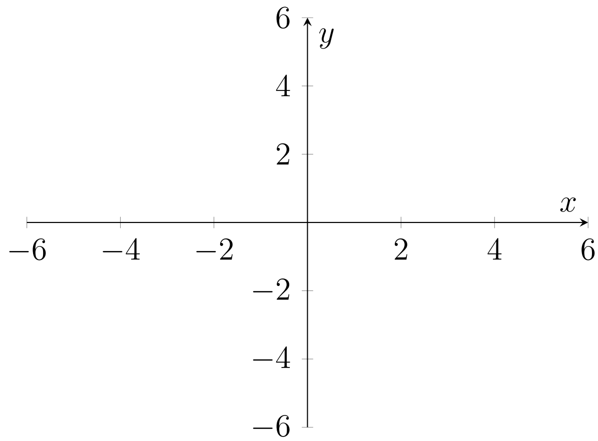


Consider the function

$$h : (-\infty, 0) \to \mathbb{R}, h(x) = \frac{x^3 + 1}{2x^2}$$

d) Find the rule of $h^{-1}(x)$ and state its maximal domain.

e) Sketch the graph of $y = h^{-1}(x)$ over its maximal domain. Labelling all axes intercepts with their coordinates and all asymptotes with their equation.



f) Find the value of $h^{-1}(1)$.

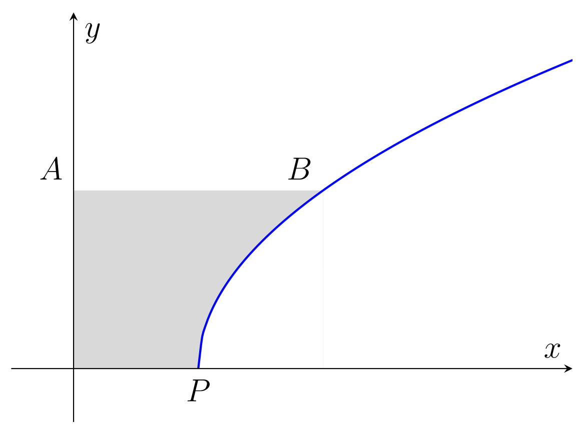
Question 41

a) Evaluate the integral $\displaystyle\int_1^2 x\sqrt{x-1}\,dx$.

b) Solve the equation $x\sqrt{x-1} = 2$ for x, where $x \in \mathbb{R}$.

c) The curve C has the equation $y^2 = x\sqrt{x-1}$. Part of this curve is shown in the diagram below along with the horizontal line $y = \sqrt{2}$. Point A has coordinates $(0, \sqrt{2})$, point B represents the intersection point of curve C and the line $y = \sqrt{2}$, while point P represents the x-intercept of curve C.

The region bounded by C, the line $y = \sqrt{2}$ and the coordinates axes is rotated about the x-axis to form a solid of revolution. Find the volume of this solid in the form $k\pi$, where you should state the exact value of k.



Question 42

a) Find the values of A and B such that $3x + 1 = A(2x + 2) + B$.

b) Hence, express $\displaystyle\int_{-1}^{0} \frac{3x+1}{x^2+2x+2}\,dx$ in the form $a + b\log_e 2$, where a and b are exact real constants to be found.

Question 43

Find $\displaystyle\int_{0}^{1} \frac{3x+1}{x^2+2x+1}\,dx$ in the form $m + \log_e n$, where m and n are exact values to be found.

Question 44

Use calculus to evaluate $\displaystyle\int_{-1}^{0} \frac{3x}{x^2+x-2}\,dx$.

Question 45

a) By using substitution $u = e^{2x} - 1$ show that

$$\int (e^{4x} + 3e^{2x})\sqrt{e^{2x}-1}\,dx = \frac{1}{2}\int u\sqrt{u} + 4\sqrt{u}\,du.$$

b) Hence, express $\displaystyle\int (e^{4x} + 3e^{2x})\sqrt{e^{2x}-1}\,dx$ in the form $(pe^{2x}+q)(e^{2x}-1)^{\frac{3}{2}}$, where $p, q \in \mathbb{R}$.

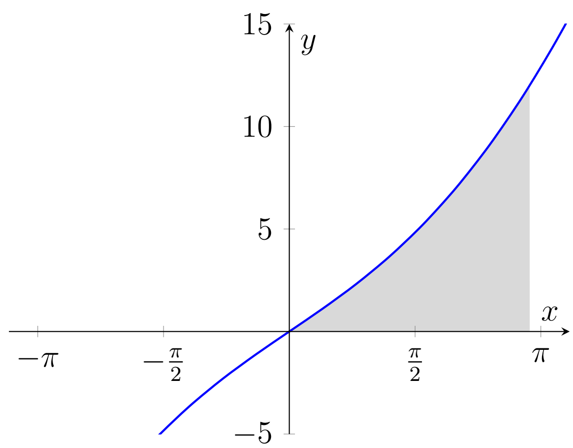
Question 46

Find the volume generated when the region enclosed by the curve $y = \dfrac{1}{\sqrt{1-x^2}}$ the x-axis, the y-axis and the line $x = \dfrac{1}{2}$ is rotated about the x-axis to form a solid of revolution. Give your answer in the form $\dfrac{\pi}{2}\log_e A,\ A \in \mathbb{Z}$.

Question 47

Find the area of the region bounded by the x-axis, the line $x = 3$ and the function $y = x\sqrt{x^2+7}$ and express your answer in the form

$$\frac{1}{m}\left[n^m - \sqrt{(m+n)^m}\right],\ m, n \in \mathbb{R}$$



$m =$

$n =$

Question 48

a) Differentiate $x\cos^{-1}(\frac{x}{2})$.

b) Hence, find an anti-derivative of $\cos^{-1}(\frac{x}{2})$.

c) Use your answer from part b. to show that the exact area enclosed by the graph of $y = \cos^{-1}(\frac{x}{2})$, the x-axis and the lines $x = 1$ and $x = 2$ is $\sqrt{3} - \frac{\pi}{3}$.

Question 49

Let $U_n = \displaystyle\int_0^{\frac{\pi}{4}} \tan^n(x)\ dx$, where $n \in \mathbb{Z}$ and $z > 1$.

a) Express $U_n + U_{n-2}$ in terms of n.

b) Hence show that $U_6 = \dfrac{13}{15} - \dfrac{\pi}{4}$.

Question 50

Gordon Ramsay is baking a cake in a tin with the base and top being circular. Its radii are 12cm and 4cm respectively and the height of the tin is 30cm.

The arc measuring the edge of the tin is modelled by the equation

$$y = \frac{a}{x^2} + b.$$

a) Show that the values of a and b are 540 and $-\frac{15}{4}$

b) Find the volume of the cake tin (correct to 1 decimal place)

c) Show that the volume of the cake is: $540\pi \left(\log_e \left(\frac{4h+15}{15} \right) \right)$ cm^3.

Unfortunately the cake has a crack in the bottom and the batter leaks out at a rate proportional to the square root of the remaining height. Initially when the cake tin is filled with batter to a height of 25cm, this rate is 15 cm^3/sec.

d) Show that the definite integral that gives the time, in minutes, for the cake tin to empty is

$$t = 12\pi \int_0^{25} \frac{1}{\sqrt{h}(4h+15)}\, dh$$

e) Hence, find the time, in minutes, for the cake tin to empty (correct to 1 decimal place)

f) Show that $\dfrac{d}{dx}\left[\tan^{-1}\left(2\sqrt{\dfrac{x}{15}}\right)\right] = \dfrac{\sqrt{15}}{\sqrt{x}(4x+15)}$

g) Hence, show that the exact time, in minutes, for the cake tin to empty is $\dfrac{4\pi\sqrt{15}}{5}\tan^{-1}\left(\dfrac{2\sqrt{15}}{3}\right)$.

Question 51

Find an anti-derivative of $\dfrac{2+6x}{\sqrt{4-x^2}}$.

Question 52

a) Expand $\dfrac{2x^2-x+4}{x^3+4x}$ into partial fractions. Ensure that you show appropriate working.

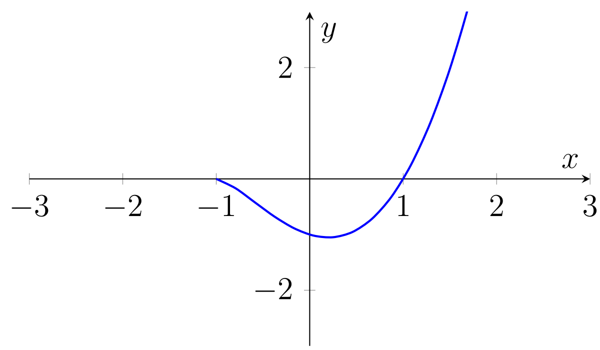
b) Hence determine $\displaystyle\int \frac{2x^2-x+4}{x^3+4x}\,dx$.

Question 53

Find an anti-derivative of $\dfrac{x+2}{1+x^2}$.

Question 54

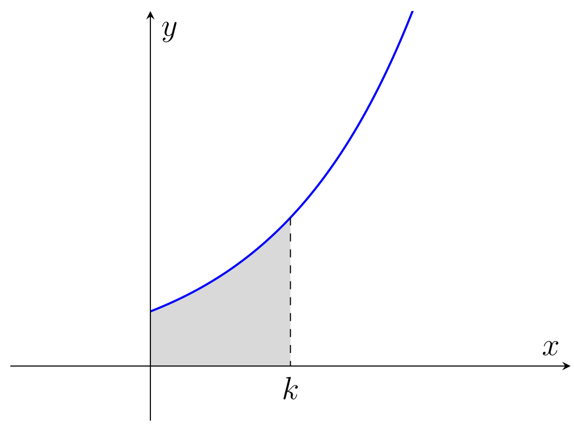
Part of the graph with equation $y = (x^2-1)\sqrt{x+1}$ is shown below.



Find the area that is bounded by the curve and the x-axis. Give your answer in the form $\dfrac{a\sqrt{b}}{c}$ where a, b and c are integers.

Question 55

Let $f(x) = e^{2x}$. Let R be the region in the first quadrant bounded by the graph of f, the coordinate axes and the vertical line $x = k$, where $k > 0$, as shown below.



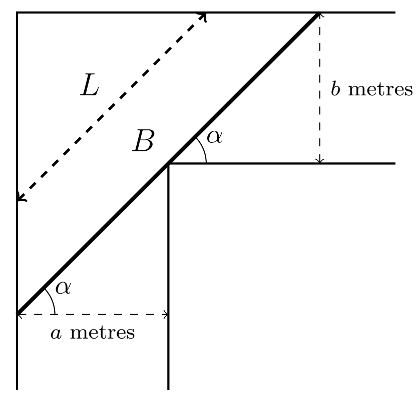
a) Write down, but do not evaluate, an expression involving an integral that gives the perimeter of R in terms of k.

b) The region r is rotated about the x-axis to form a solid. Find the volume V, of the solid in terms of k.

c) The volume,V , found in part b. changes as k changes. If $\dfrac{dk}{dt} = \dfrac{1}{3}$, determine $\dfrac{dV}{dt}$ when $k = \dfrac{1}{2}$.

Question 56

A pipe of negligible radius is carried horizontally around a corner from a hallway a metres wide into a hallway b metres wide as shown in the diagram.
L is the length of the pipe in metres and α is the angle that the pipe makes with one passage.



a) Show that $L = b\csc(\alpha) + a\sec(\alpha)$.

b) Find $\dfrac{dL}{d\alpha}$

c) Show that L is a maximum when $\alpha = \tan^{-1}\left(\left(\dfrac{b}{a}\right)^{\frac{1}{3}}\right)$

d) For this value of α, express:

i. $\operatorname{cosec}(\alpha)$ in terms of a and b only.

ii. $\sec(\alpha)$ in terms of a and b only.

e) Hence find the maximum length of pipe in fully simplified form, in terms of a and b only, that can be carried horizontally around the corner.

f) If $a = 4$ and $b = 8$, find the maximum length of pipe in metres correct to two decimal places.

Question 57

The region in the first quadrant enclosed by the coordinate axes, the graph with equation $y = \dfrac{1}{\sqrt{1+9x^2}}$ and the straight line $x = a$ where $a > 0$ is rotated about the x-axis to form a solid of revolution.

a) Express the volume of the solid of revolution as a definite integral.

b) Calculate the volume of the solid of revolution, in terms of a.

c) Find the exact value of a if the volume is $\dfrac{\pi^2}{9}$ cubic units.

Question 58

The following uses the technique of *integration by parts* to integrate

$$\int \sin^2(x)\ dx$$

a) Without using integration by parts determine $\int \sin^2(x)\ dx$, and express your answer in the form $\frac{1}{p}[qx - \sin(qx)]$, where p and q are constants.

b) Use the technique of integration by parts twice to determine $\int \sin^2(x)\ dx$ by initially setting $u = \sin^2(x)$ and $v' = 1$.

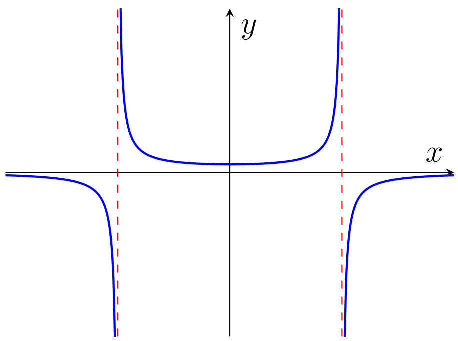
c) Using integration by parts once to determine $\int \sin^2(x)\ dx$ by setting $u = \sin(x)$ and $v' = \sin(x)$ to achieve the same answer you obtained in part a.

Multiple Choice Questions

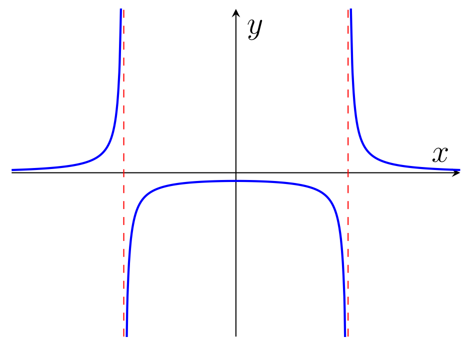
Question 1

For the function $f(x) = \dfrac{1}{ax^2 + bx + c}$, if $b^2 - 4ac > 0$ and $a < 0$, then the graph of $y = f(x)$ is best represented by one of the following.

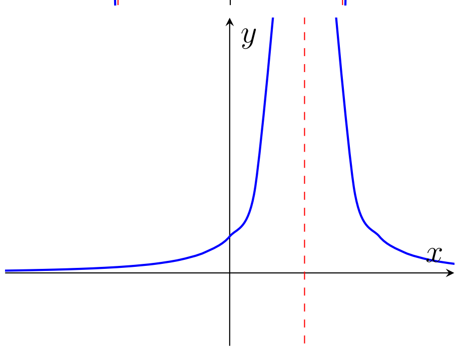
A.



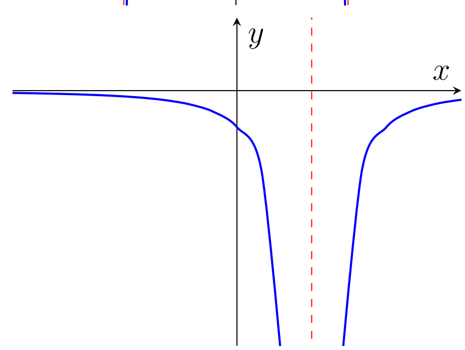
B.



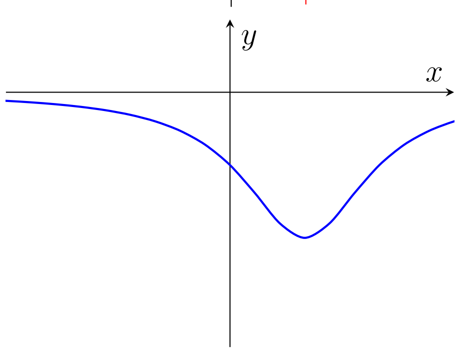
C.



D.



E.



Question 2

The graph of $y = \dfrac{x^2 + a^2}{x^2}$ has:

A. a single asymptote at $x = 0$ and no turning points.

B. two asymptotes, $x = 0$ and $y = 0$, and no turning points.

C. two asymptotes, $x = 0$ and $y = 1$, and no turning points.

D. a single asymptote at $x = 0$ and turning points at $x = \pm\sqrt{a}$.

E. a single asymptote at $x = 0$ and turning points at $x = \pm a$.

Question 3

An empty container has the shape of an inverted cone with base diameter equal to its height. Water is poured into the container at the rate of 10000 cm^3 per minute. The rate at which the water level is rising, in cm/min, when the water level is 500 cm above the vertex of the cone is:

A. $\dfrac{16}{\pi}$

B. $\dfrac{4}{\pi}$

C. $\dfrac{1}{\pi}$

D. $\dfrac{16}{3\pi}$

E. $\dfrac{24}{25\pi}$

Question 4

The graph of $y = \dfrac{-x^2+1}{2x}$ may have some of the following features.

A. no straight line asymptotes

B. $y = 2x$ as its only straight line asymptote

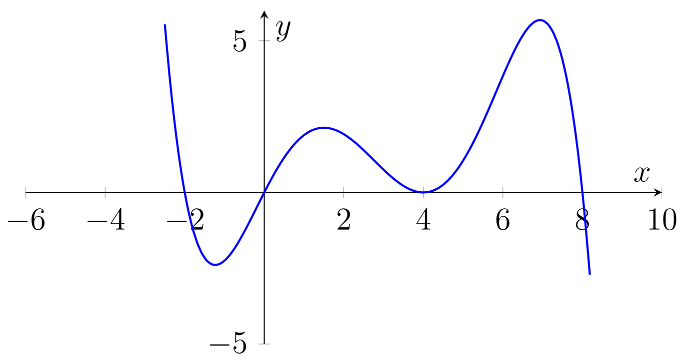
C. $x = 0$ as its only straight line asymptote

D. $y = 0$ and $y = -\dfrac{x}{2}$ as its straight line asymptotes

E. $x = 0$ and $y = -\dfrac{x}{2}$ as its straight line asymptotes

Question 5

The graph of $y = f(x)$ is shown below.
If $F(x)$ is the anti-derivative of $f(x)$ then the graph of $y = F(x)$ has a:



A. Stationary point at $x = 0$ only.

B. Stationary point at $x = -1, 2, 4$ and 7.

C. Stationary point of inflection at $x = 4$ only.

D. Stationary point of inflection at $x = -1, x = 2, x = 4$ and $x = 7$.

E. Stationary point of inflection at $x = -2, x = 0, x = 4$ and $x = 8$.

Question 6

The curve C has equation $y = \dfrac{2ax^2 + 2bx + 7}{2x + 5}$, where $x \neq -\dfrac{5}{2}$ and a, b are non-zero constants. The oblique asymptote cuts the x-axis at (-1,0). The relationship between a and b is given by:

A. $b = 4a$

B. $7a = 2b$

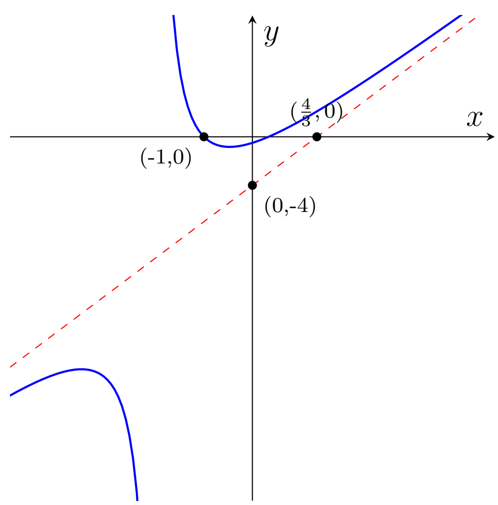
C. $a = 4b$

D. $2a = 7b$

E. $a = b$

Question 7

The equation of the graph on the right is:



A. $y = \dfrac{x^2 + 2x + 1}{3x + 2}$

B. $y = \dfrac{x^2 - 2x + 1}{3x - 2}$

C. $y = \dfrac{3x^2 + 2x - 1}{x + 2}$

D. $y = \dfrac{3x^2 - 2x - 1}{2x + 2}$

E. $y = \dfrac{3x + 2}{x - 4}$

Question 8

If $y = \sin^{-1}\left(-\frac{1}{x}\right)$ then $\frac{dy}{dx}$ is equal to:

A. $\frac{1}{x\sqrt{x^2-1}}, |x| > 1$

B. $\frac{-1}{x\sqrt{x^2-1}}, |x| > 1$

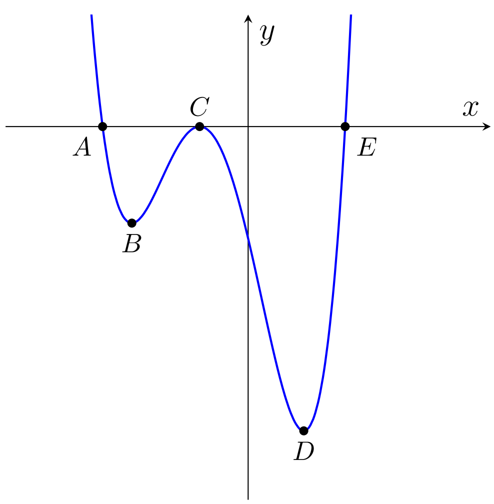
C. $\frac{-1}{x\sqrt{x^2-1}}, |x| < 1$

D. $\frac{1}{x\sqrt{1-x^2}}, |x| < 1$

E. $\frac{1}{\sqrt{x^2-1}}, |x| > 1$

Question 9

Part of the graph of $y = f'(x)$ is shown below.
Which of the following statements is true about the graph of $y = f(x)$?



A. B and D represents stationary points of inflection.

B. A and E represent non-stationary points of inflection.

C. C is a stationary point of inflection.

D. A is a local minimum stationary point.

E. E is a local maximum stationary point.

Question 10

If $y = \sin^{-1}\left(\frac{1}{x^2}\right)$ and $x \geq 1$ then $\frac{dy}{dx}$ us equal to:

A. $\frac{-2x}{\sqrt{x^4 - 1}}$

B. $\frac{2x}{\sqrt{x^4 - 1}}$

C. $\frac{-2}{x\sqrt{x^4 - 1}}$

D. $\frac{2}{x\sqrt{x^4 - 1}}$

E. $\frac{-2}{x^3\sqrt{1 - x^2}}$

Question 11

A function, $y = f(x)$, has the following properties at a point where $x = a$, $f'(a) = 0$, $f''(a) = 0$ and $f''(a) > 0$ if $x > a$. Also $f''(x) = 0$ when $x < a$. Which of the following is true about $y = f(x)$ at the point where $x = a$.

A. $f(x)$ has a local minimum point at $x = a$.

B. $f(x)$ has no inflection points at $x = a$ since $f''(x) \neq 0$.

C. $f(x)$ has a non-stationary point of inflection at $x = a$.

D. $f(x)$ has a stationary point of inflection at $x = a$.

E. $f(x)$ has a local maximum point at $x = a$.

Question 12

With a suitable substitution $\int_0^{\frac{\pi}{3}} \cos^2(3x)\sin^3(3x)\ dx$ can be expressed as:

A. $3\int_{-1}^{1} u^2(u^2-1)\ du$

B. $3\int_{-1}^{1} u^2(1-u^2)\ du$

C. $\frac{1}{3}\int_{-1}^{1} u^2(u^2-1)\ du$

D. $\frac{1}{3}\int_{-1}^{1} u^2(1-u^2)\ du$

E. $\frac{1}{3}\int_{0}^{\frac{\pi}{3}} u^2(1-u^2)\ du$

Question 13

The area bounded by the curves $f(x) = \dfrac{e^x}{e^x+1}$ and $g(x) = e^{-x}$, for $\dfrac{1}{2} \leq x \leq 1$ is given by:

A. $\int_{\frac{1}{2}}^{1} \dfrac{e^x+1-e^{2x}}{e^{2x}+e^x}\ dx$

B. $\int_{\frac{1}{2}}^{1} \dfrac{e^x-1+e^{2x}}{e^{2x}+e^x}\ dx$

C. $\int_{\frac{1}{2}}^{1} \dfrac{e^{2x}-1-e^x}{e^{2x}+1}\ dx$

D. $\int_{\frac{1}{2}}^{1} \dfrac{e^{2x}-1-e^x}{e^{2x}+e^x}\ dx$

E. $\int_{\frac{1}{2}}^{1} \dfrac{1-e^x-e^{2x}}{e^{2x}+1}\ dx$

Question 14

The graph of $g(x)$ is given by:

$$g : [0, 1] \to \mathbb{R},\ \text{where } g(x) = \cos^{-1}(x)$$

The region bounded by $y = g(x)$ the positive x-axis and y-axis is rotated about the y-axis to form a solid of revolution. The volume of this solid in cubic units is given by:

A. $\dfrac{\pi}{2} \displaystyle\int_0^{\frac{\pi}{2}} (\cos 2y + 1)\ dy$

B. $\pi \displaystyle\int_0^{\frac{\pi}{2}} (\cos 2y + 1)\ dy$

C. $\pi \displaystyle\int_0^{\frac{\pi}{2}} \cos(y)\ dy$

D. $\pi \displaystyle\int_0^{1} \cos^{-1}(x^2)\ dy$

E. $\pi \displaystyle\int_0^{1} (\cos^{-1}(x))^2\ dy$

Question 15

$\displaystyle\int_0^{m} \tan(x)\sec^2(x)\ dx = \frac{3}{2}$ where $m \in \left(0, \dfrac{\pi}{2}\right)$. The value of m is:

A. $\dfrac{1}{2}$

B. 1

C. $\dfrac{\pi}{3}$

D. $\dfrac{\pi}{6}$

E. $\dfrac{\pi}{8}$

Question 16

$\int_0^{\frac{\pi}{6}} \cos(x)\sin^3(x)\ dx$ as an integral with respect to u, where $u = \sin(x)$ is:

A. $\int_{\frac{1}{2}}^{0} u^3\ du$

B. $\int_0^{\frac{\pi}{3}} u^3\ du$

C. $\int_0^{\frac{1}{2}} u^3\sqrt{1-u^2}\ du$

D. $\int_{\frac{1}{2}}^{0} u^3\sqrt{1-u^2}\ du$

E. $\int_0^{\frac{1}{2}} u^3\ du$

Question 17

An anti-derivative of $\frac{\sin(x)}{\cos^3(x)}$ is:

A. $\frac{1}{\cos^4(x)}$

B. $\frac{-1}{\cos^2(x)}$

C. $\frac{-1}{\sin^2(x)}$

D. $\frac{1}{2\cos^2(x)}$

E. $\frac{1}{4\cos^2(x)}$

Question 18

The area between the graphs of $y = x^3$ and $y = 4x$ is given by:

A. $\displaystyle\int_{-2}^{2} 4x - x^3 \; dx$

B. $\displaystyle\int_{-2}^{2} x^3 - 4 \; dx$

C. $\displaystyle\int_{-2}^{0} 4x - x^3 \; dx + \int_{0}^{2} 4x - x^3 \; dx$

D. $\displaystyle -\int_{-2}^{0} x^3 - 4 \; dx + \int_{0}^{2} x^3 - 4 \; dx$

E. $\displaystyle -\int_{-2}^{0} 4x - x^3 \; dx + \int_{0}^{2} 4x - x^3 \; dx$

Question 19

A spherical balloon of radius r metres is leaking at a constant rate of 2 m^3/h. If the balloon retains its shape, the rate at which the radius is decreasing when the radius is 3 metres is equal to:

A. $\dfrac{1}{18\pi}$ m/h

B. 36π m/h

C. $\dfrac{1}{36\pi}$ m/h

D. 72π m/h

E. $-\dfrac{1}{6\pi}$ m/h

Question 20

With a suitable substitution $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\cos(x)}{\sin(x)+\cos^2(x)}\,dx$ can be expressed as:

A. $\displaystyle\int_0^{\frac{\pi}{4}} (\frac{1}{1+u-u^2})\,du$

B. $\displaystyle\int_0^{\frac{1}{\sqrt{2}}} (\frac{1}{u^2-u-1})\,du$

C. $\displaystyle\int_0^{\frac{\pi}{4}} (\frac{1}{u^2+u-1})\,du$

D. $\displaystyle\int_0^{\frac{1}{\sqrt{2}}} (\frac{1}{u^2+u-1})\,du$

E. $\displaystyle\int_0^{\frac{1}{\sqrt{2}}} (\frac{1}{1+u-u^2})\,du$

Question 21

The area bounded by the curve $\dfrac{2x}{(x^2-4)^2}$, the x-axis, and the lines $x = \pm 1$ is given by:

A. $2\displaystyle\int_0^1 f(x)\,dx$

B. $\displaystyle\int_{-1}^1 f(x)\,dx$

C. $\pi\displaystyle\int_{-1}^1 (f(x))^2\,dx$

D. $2\displaystyle\int_{-1}^0 f(x)\,dx$

E. $2\displaystyle\int_{-1}^1 f(x)\,dx$

Question 22

If $y = \tan^{-1}(u+1)$ and $u = (x+1)^{-1}$, then $\dfrac{dy}{dx}$ at $x = -2$ is:

A. 1

B. 2

C. -2

D. $-\dfrac{1}{2}$

E. -1

Question 23

An anti-derivative of $\dfrac{2}{\sqrt{1-16x^2}}$ is:

A. $\sin^{-1}(\dfrac{x}{4})$

B. $\dfrac{1}{2}\sin^{-1}(\dfrac{x}{4})$

C. $\sin^{-1}(4x)$

D. $\dfrac{1}{2}\sin^{-1}(4x)$

E. $\dfrac{1}{8}\sin^{1}(4x)$

Question 24

If $\frac{d}{dx}(f(x)\sin(x)) = g(x) + \cos(x)$ and $f(\frac{\pi}{2}) = -1$, and $f(0) = 2$, then $\int_0^{\frac{\pi}{2}} g(x)\ dx$ is equal to:

A. -1

B. 0

C. 2

D. -2

E. 1

Question 25

The volume of the solid of revolution formed by rotating the graph of $y = \sqrt{9-(x-1)^2}$ about the x-axis is given by:

A. $\pi\int_{-2}^{4}(\sqrt{9-(x-1)^2})\ dx$

B. $\pi\int_{2}^{4}(\sqrt{9-(x-1)^2})\ dx$

C. $4\pi(3)^2$

D. $\pi\int_{-3}^{3}(\sqrt{9-(x-1)^2})\ dx$

E. $\pi\int_{-2}^{4}(\sqrt{9-(x-1)^2})^2\ dx$

Question 26

$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\sin(2x)}{2+\cos(2x)}\,dx$ is equal to:

A. $\frac{1}{\sqrt{2}}$

B. $\log_e \frac{1}{\sqrt{2}}$

C. $\log_e 2$

D. $\frac{1}{2}\log_e 2$

E. 1

Question 27

The graph of the function with the rule $f(x) = \frac{1}{(x-4)(x+2)}$ over its maximal domain has:

A. asymptotes $x = -4$ and $x = 2$ and a turning point at (1,-5)

B. asymptotes $x = -4$ and $x = 2$ and a turning point at $(1, -\frac{1}{5})$

C. asymptotes $x = 4$ and $x = 2$ and a turning point at $(1, -\frac{1}{9})$

D. asymptotes $x = 4$ and $x = 2$ and a turning point at $(1, \frac{1}{9})$

E. asymptotes $x = -4$ and $x = 2$ and a turning point at $(1, -\frac{1}{9})$

Question 28

An anti-derivative of $\tan^2(x)$ is:

A. $\tan(x)\sec^2(x)$

B. $(\log_e(\cos(x)))^2$

C. $\tan(x) - x$

D. $\dfrac{1-\cos^2(x)}{\cos^2(x)}$

E. None of these

Question 29

The graph of $y = \dfrac{1}{x^2+bx+c}$ has two distinct vertical asymptotes if:

A. $b = 0$

B. $c < b$

C. $b = c$

D. $b^2 > 4c$

E. $b^2 = 4c$

Question 30

Using a suitable substitution, $\displaystyle\int_0^2 \frac{2x-1}{\sqrt{3-x}}\,dx$ can be expressed as:

A. $\displaystyle\int_0^2 (7-2u)\,du$

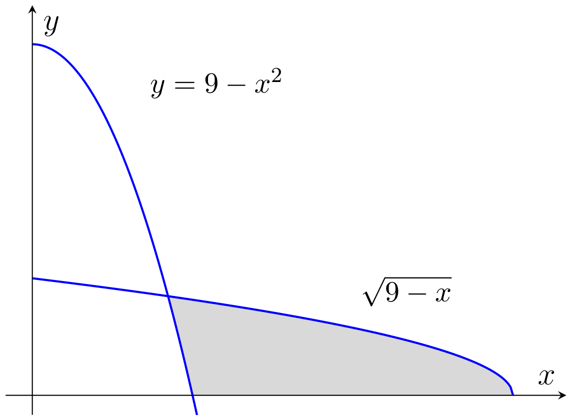
B. $\displaystyle\int_0^2 (5u^{-\frac{1}{2}} - 2u^{\frac{1}{2}})\,du$

C. $\displaystyle\int_1^3 (7u^{-\frac{1}{2}} - 2u)\,du$

D. $\displaystyle-\int_1^3 (5u^{-\frac{1}{2}} - 2u^{\frac{1}{2}})\,du$

E. $\displaystyle\int_1^3 (5u^{-\frac{1}{2}} - 2u^{\frac{1}{2}})\,du$

Question 31



The area of the shaded region above can be found by evaluating :

A. $\displaystyle\int_2 .54^3 (9 - x^2)\ dx + \int_3^9 (\sqrt{9-x})\ dx$

B. $\displaystyle\int_2 .54^3 (9 - x^2 - \sqrt{9-x})\ dx + \int_3^9 (\sqrt{9-x})\ dx$

C. $\displaystyle\int_{2.54}^3 (\sqrt{9-x})\ dx + \int_{2.54}^3 (\sqrt{9-x} - 9 + x^2)\ dx$

D. $\displaystyle\int_3^9 (\sqrt{9-x})\ dx + \int_{2.54}^3 (\sqrt{9-x} - 9 + x^2)\ dx$

E. $\displaystyle\int_{2.54}^9 (\sqrt{9-x})\ dx + \int_{2.54}^3 (\sqrt{9-x} - 9 + x^2)\ dx$

Question 32

An anti-derivative of $\dfrac{\cos^3(x)}{\sin^2(x)}$ is:

A. $\cos(x) + \csc(x)$

B. $\dfrac{3\cos^2(x)}{2\sin(x)}$

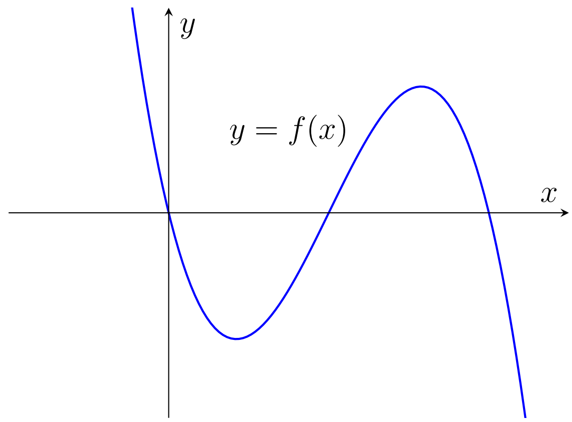
C. $\dfrac{3\sin^4(x)}{4\cos^3(x)}$

D. $\cos(x) - \sin(x)$

E. $-\sin(x) - \csc(x)$

Question 33

The graph of $y = f(x)$ is shown below.



The anti-derivative of $f(x)$ has the rule $y = F(x)$. For $F(x)$ it is true that it:

A. has no points of inflection.

B. has no stationary points of inflection.

C. has no stationary points.

D. has two stationary points.

E. has one local maximum and one local minimum.

Question 34

$\displaystyle\int_{\frac{\pi}{2}}^{m} 2\cot(x)\csc^2(x)\ dx = -1$, where $m \in \left(\frac{\pi}{2}, \pi\right)$. The value of m is:

A. 0.5

B. 1

C. $\dfrac{\pi}{3}$

D. $\dfrac{3\pi}{4}$

E. $\dfrac{\pi}{4}$

Question 35

The area bounded by the curve $y^2 = 4x$ and the line $y = x$ in the first quadrant is rotated about the x-axis. The volume of the solid generated is given by:

A. $y = \pi \int_0^4 (x^2 - 4x)\ dx$

B. $y = \pi \int_0^4 (4x - 2\sqrt{x})^2\ dx$

C. $y = \pi \int_0^4 (y^2 - \frac{y^4}{16})\ dy$

D. $y = \pi \int_0^4 (y^2 - \frac{y^2}{4})^2\ dy$

E. $y = \pi \int_0^4 (4x - x^2)\ dx$

Question 36

$\int_4^a \frac{1}{x^2 - 9}\ dx = \frac{1}{6} \log_e 3$, where $a > 4$.

The value of a is:

A. 2

B. 2.25

C. 7

D. 7.5

E. 8.25

Question 37

An appropriate substitution can convert the integral $\displaystyle\int_a^{2a} \frac{f(x+a)}{(x-a)^2}\,dx$ into:

A. $\displaystyle\int_{-a}^{0} \frac{f(u+2a)}{u^2}\,du$

B. $\displaystyle\int_{0}^{a} \frac{f(u+2a)}{u^2}\,du$

C. $\displaystyle\int_{2a}^{3a} \frac{f(u-2a)}{u^2}\,du$

D. $\displaystyle\int_{-a}^{a} \frac{f(u+2a)}{u^2}\,du$

E. $\displaystyle\int_{0}^{a} \frac{f(u)}{(u-2a)^2}\,du$

Question 38

If $F'(x) = f(x)$, then an anti-derivative of $3f(3-2x)$ is:

A. $\dfrac{3}{2}F(3-2x)$

B. $-\dfrac{3}{4}F(3-2x)^2$

C. $\dfrac{3}{4}F(3-2x)^2$

D. $-\dfrac{3}{2}F(3-2x)$

E. $-\dfrac{3}{2}f(3-2x)$

Question 39

The expression $\int_0^{\pi} 2x\cos(x^2)\ dx$ is equal to:

A. $\cos(\pi)$

B. $\sin(\pi^2)$

C. 0

D. 1

E. 2

Question 40

Using an appropriate substitution, the integral $\int_{-1}^{1} (2x-1)\sqrt{4-3x}\ dx$ can be written as:

A. $\frac{2}{9}\int_1^{\sqrt{7}} u(5-2u^2)\ du$

B. $-\frac{2}{9}\int_1^{\sqrt{7}} u^2(5-2u^2)\ du$

C. $-\frac{2}{3}\int_1^{\sqrt{7}} u^2(5-2u^2)\ du$

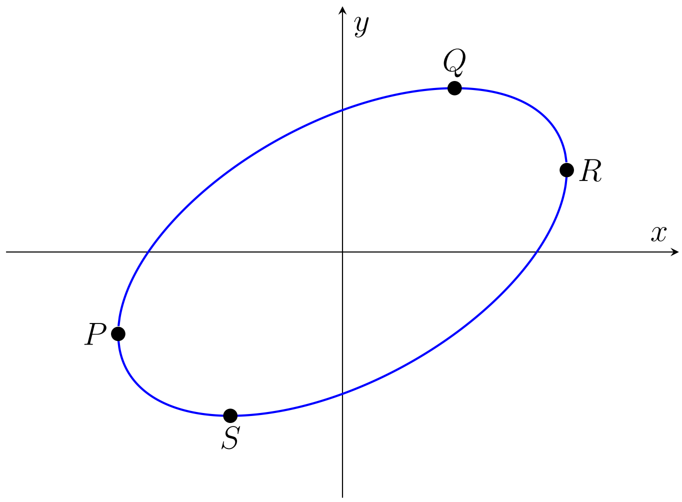
D. $\frac{2}{9}\int_1^{\sqrt{7}} u^2(5-2u^2)\ du$

E. $\frac{2}{3}\int_1^{\sqrt{7}} u^2(5-2u^2)\ du$

Extended Response Questions

Question 1

The curve C, is called a degenerate ellipse, is defined by the equation $x^2 - xy + y^2 = 3$. A sketch of the curve is shown below.



The points Q and S on the curve C have horizontal tangents, while points P and R on the curve C have vertical tangents.

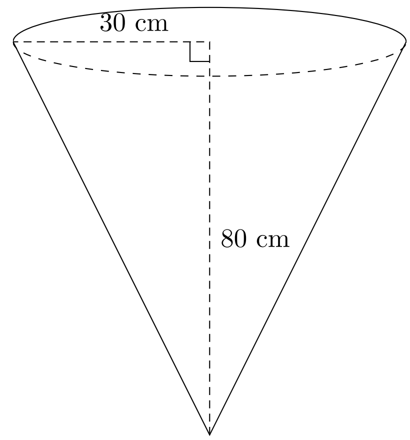
a) Show that for curve C, $\dfrac{dy}{dx} = \dfrac{y - 2x}{2y - x}$.

b) Find the coordinates of Q and S.

c) By considering $\dfrac{dy}{dx}$ or otherwise, find the coordinates of points P and R.

Question 2

A conical shaped urn with an open top and a tap at the base is used for cordial.
The height is 80 cm and the radius is 30cm.
Initially, 500ml of syrup is mixed with 1 litre of water.



a) Find the ratio of syrup to water.

b) Find the initial depth of the solution (to two decimal places).

A pleasant tasting cordial has a syrup to water ratio of 1:20.
Fresh water is added to the urn at a rate of 50 cm^3 per second until this ratio is reached.

c) Find the rate that the solution is rising when the depth is 40 cm.

d) Find how long it takes for the solution to reach the pleasant tasting ratio.

Once this concentration is reached, the cordial is drained out through the tap at the bottom of the urn as $50\text{cm}^3/\text{sec}$ while fresh water is continuously added at the same rate.

e) Show that the rate of change of the syrup, S, in the solution is given by $\dfrac{dS}{dt} = -\dfrac{S}{210}$, where t is the time in seconds after the ratio of 1:120 is reached.

f) Use calculus to find the amount of syrup in the solution at any time t minutes after the solution becomes pleasant tasting.

g) When there is less than 20cm^3 of syrup in the solution the cordial is too weak and more syrup is added. Find how long it will take until more syrup is needed? (correct to the nearest minute).

Question 3

Mr. Jelinek creates a new shape for a birthday cake. The shape is formed when the curve given by the equation: $y = 5\cos^{-1}\left(\frac{x}{10}\right)$ for $x \in [0, 10]$ is rotated about the y-axis to form a solid of revolution. All units are in centimetres.

a) Set up an integral expression to find the volume of the cake.

b) Calculate the volume of the cake.

Consider the cross section of the cake, (half of which is shown in the graph above)

A candle is inserted where $y = \frac{5\pi}{3}$.

c) Find the gradient of the cake at the point where the candle is inserted.

d) The cake is cut from the center out to the edge of the cake.
Use calculus to find the cross-sectional area of a piece of cake.

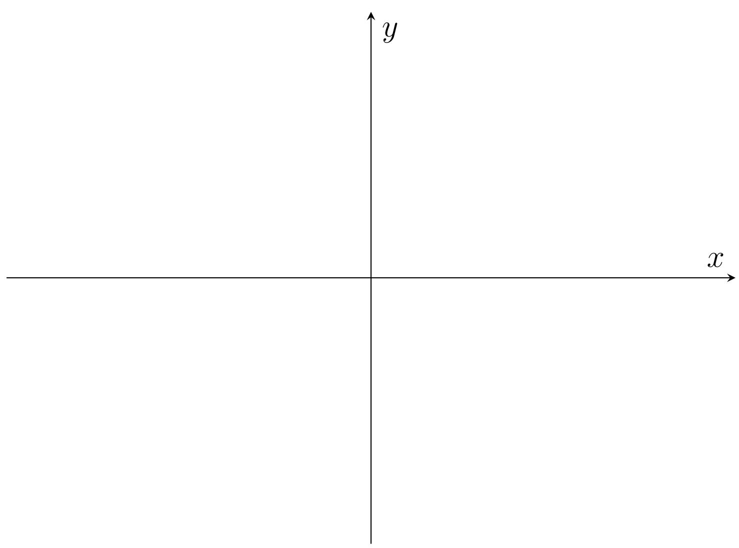
Question 4

Consider the function $f(x) = \dfrac{x^3 + 2x^2 - 4}{2x^2}, x \in \mathbb{R}\backslash\{a\}$

a) State the value of a.

b) Find the coordinates of any stationary points and state their nature.

c) Graph $y = f(x)$ on the axes below, labelling all axes intercepts (correct to two decimal places), stationary points and asymptotes.



d) Find $F(x)$, the anti-derivative of $f(x)$.

e) Hence, find the integer value of b such that the area between the curves $g(x) = \frac{1}{2}x + 1$ and $h(x) = \frac{2}{x^2}$ and the lines $x = 2$ and $x = b$ is $\frac{265}{2}$ units^2.

Question 5

The section of the straight line with equation $y = \frac{x}{2} - 2$ between $x = 4$ and $x = 6$, is rotated about the y-axis to form a container with a base on the x-axis.

a) **i.** Determine an integral which gives the volume of the container.

ii. Find the exact volume of the container.

b) **i.** If the container is filled to $\frac{3}{4}$ of its height with water, what is the exact volume of water in the container ?

ii. What is the exact area of the water surface when the container is filled with this volume of water.

c) **i.** If the container is filled to a height of h metres, show that an expression for the volume of water in the container is $V = \pi\left(\frac{4h^3}{3} + 8h^2 + 16h\right)$.

ii. If half the volume of the container is filled with water, find the height of water in the container. Give your answer correct to 3 decimal places.

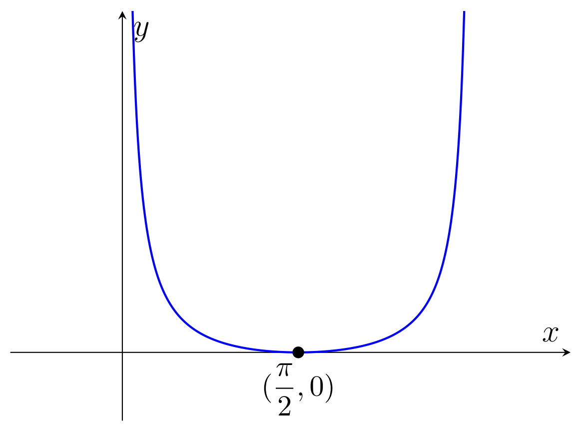
d) If the container is filled at a rate of 0.2 cubic units per minute, at what exact rate is the height of the water in the container increasing when $h = 0.5$?

Question 6

a) The vertical cross-section of a water tank is modelled by the curve with equation

$$y = a\csc(x) - b$$

The curve passes through the points $\left(\frac{\pi}{3}, 2\sqrt{3} - 3\right)$ and $\left(\frac{\pi}{2}, 0\right)$.
Find the values of a and b.



a) The vertical cross-section of a different tank is modelled by the curve $y = 2\csc(x) - 2$ and the curve passes through the points $(\frac{\pi}{6}, 2)$ and $(\frac{\pi}{2}, 0)$. All units are in metres.

i. When the depth of water at the centre of the cross-section is 3 metres, calculate the width of the tank at water level correct to two decimal places.

ii. When the width of the tank at water level is $\frac{\pi}{2}$ metres, calculate the depth of water at the centre of the cross-section correct to two decimal places.

The maximum level of water in the tank is at a point C. C is given by the intersection of a pair of normal lines to the curve at the points M and N where $x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$ respectively. The gradient of the tangents at M and N are $-4\sqrt{3}$ and $4\sqrt{3}$ respectively.

b) Find the coordinates of C and hence the maximum depth of the water in the tank, stating your answer to two decimal places.

Question 7

Given $g(x) = e^{-x} f(x)$, where $f(x)$ is differentiable for all real numbers:

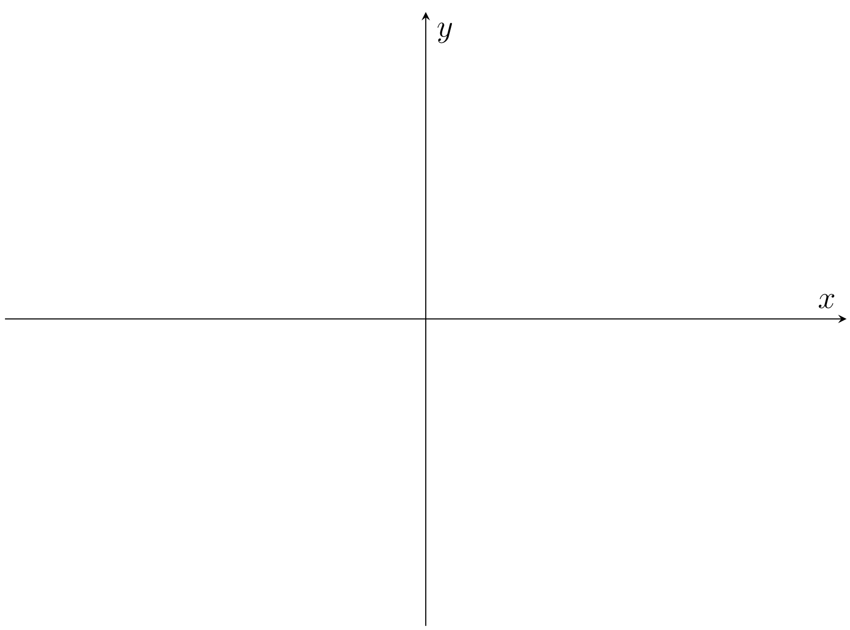
a) Find $g'(x)$.

b) Find $g''(x)$.

c) Show that $g''(x) = 0$ implies that $f(x) = 2f'(x) - f''(x)$.

d) Let $f(x) = x^2$. Then $g(x) = x^2 e^{-x}$. Using the result of part **c.**, or otherwise, find the values of x for which there are points of inflection for the graph $y = g(x)$.

e) Sketch the graph of $g(x) = x^2 e^{-x}$, clearly indicating the x-coordinate of any points of inflection and give the coordinates of any turning points.



Question 8

Given the hyperbola defined by the relation $2x^2 - y^2 - 4x + 4y - 6 = 0$, find:

a) The equation of all asymptotes.

b) The equation of the normal to the graph of the hyperbola at the point in the fourth quadrant where $x = 5$.

c) Let m be the gradient of the normal to the graph of the hyperbola at any point. Find all possible values of m.

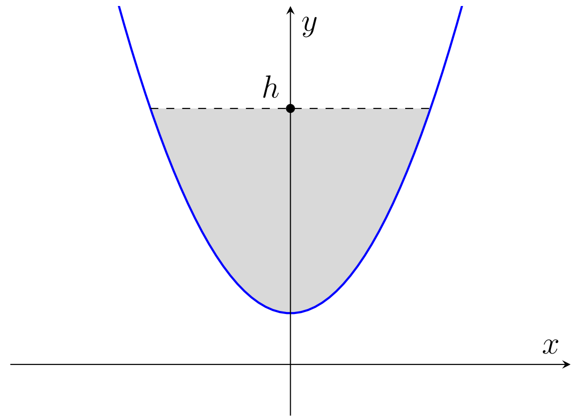
Let the hyperbola have domain $(-\infty, a] \cup [b, \infty)$ and an inverse function that passes through the points $(4, -1)$ and $(0, 3)$.

d) Find the largest possible value of a and b and state the corresponding range of the hyperbola for each part of its domain.

Question 9

The shaded region bounded by the curve with equation $y = x^2 + 4$ and line $y = h$ is rotated around the y-axis to form a solid of revolution.

a) Show that this volume v (in cm^3) in terms of h (cm) is given by $v = \pi\left(\frac{h^2}{2} - 4h + 8\right), h > 4.$



A container in the shape of this solid of revolution has water poured into it at a rate of 2 cm^3/s.

b) Find the rate (in cm/s) at which the depth of water is increasing when the depth is h cm.

No more water is added to the container once its full. Water then runs from the first container to a second container so that the depth of the first container is decreasing by 0.5 cm/s.

c) Find the rate, $\frac{dv}{dt}$, at which the volume is decreasing in terms of h.

d) The second container is such that the relationship between the volume V cm^3 and the depth of the water, H, in it is given by $V = 100H + \dfrac{1}{H} - 20$.

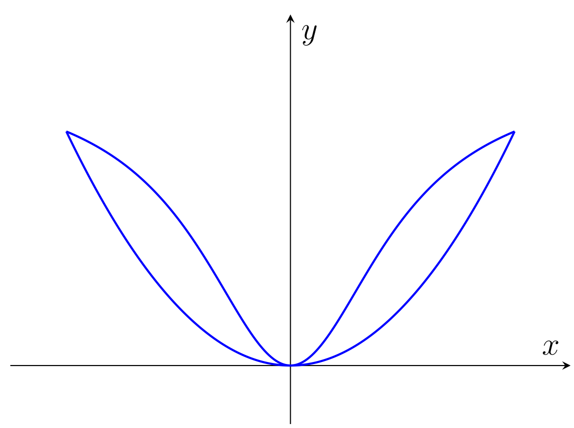
i. Find $\dfrac{dV}{dH}$ in terms of H.

ii. Find $\dfrac{dH}{dt}$ in terms of H and h.

Question 10

A company is designing a new water fountain, it is made up of an inner shell and an outer shell.

The inner shell of the water fountain is based on the volume of revolution generated by rotating the graph of $f(x)$ about the y-axis. All measurements are in metres.



$$f(x) = 1 - \frac{1}{1+x^2}, x \in [0, 2]$$

a) Let h metres be the depth of water in the fountain. Show that the volume of water in the fountain, V litres, is given by $V = 1000\pi \displaystyle\int_0^h \left(\frac{1}{1-y} - 1 \right) \, dy$.

b) Hence, find a formula for V in terms of h.

c) Find, correct to the nearest litre, the volume of water required to fill the water fountain completely.

d) Find, using CAS, the depth of water in the fountain, correct to the nearest 0.01 metre, if the volume if water in the fountain is 1800 litres.

e) To fill the fountain, water is pumped in at a constant rate of 60 litres per minute.

i. Set up a differential equation in terms of h and t.

ii. Hence, find the rate, correct to three decimal places, at which the depth of water in the fountain is increasing when the depth is 0.25 metres.

f) Plastic filler is used in the construction of the solid part of the water fountain (between the inner and outer shells).

If the outer shell is modelled by

$$h(x) = \frac{1}{5}x^2, x \in [-2, 2]$$

Find the volume of plastic filler required to create the fountain to the nearest litre.

Question 11

a) By choosing $u = x$ and $\frac{dv}{dx} = \sin(x)$, use integration by parts to show that

$$\int x\sin(x)\ dx = -x\cos(x) + \int \cos(x)\ dx$$

b) Hence, find $\int x\sin(x)\ dx$.

c) Briefly explain what happens if the choice $u = \sin(x)$ and $\frac{dv}{dx} = x$ is made instead.

Question 12

a) Using the substitution $u = \sqrt{x+4}$, show that $\displaystyle\int \frac{\sqrt{x+4}}{x}\, dx = 2\int \frac{u^2}{u^2-4}\, du$

b) Hence, show that $\displaystyle\int \frac{\sqrt{x+4}}{x} = 2\sqrt{x+4} + 2\log_e\left(\frac{|\sqrt{x+4}-2|}{|\sqrt{x+4}+2|}\right) + c.$

c) Evaluate $\displaystyle\int_{\frac{1}{2}}^{1} \frac{\sqrt{x+4}}{x}\, dx$ to 3 decimal places.

Question 13

a) Apply integration by parts to the integral $\int \sin^3(x)\ dx, u = \sin^2(x)$ and $v' = \sin(x)$ to obtain the expression:

$$\int \sin^3(x) = -\cos(x)\sin^2(x) + \int 2\sin(x)\cos^2(x)\ dx$$

b) By expressing $2\sin(x)\cos^2(x)$ in terms of $\sin(x)$ alone, show that

$$\sin^3(x)\ dx = -\frac{\sin^2(x)\cos(x)}{3} + \frac{2}{3}\int \sin(x)\ dx$$

Hence, evaluate $\sin^3(x)\ dx$ fully.

c) Hence, propose an anti-derivative of $\sin^n(x)$ of the form

$$-\frac{\sin^a(x)\cos(x)}{c} + \frac{b}{c}\int \sin^d(x)\ dx$$

where $a, b, c, d \in \mathbb{Z}\backslash\{0\}$
and hence determine a, b, c, d in terms of n. You do not need to show your working.

$a =$

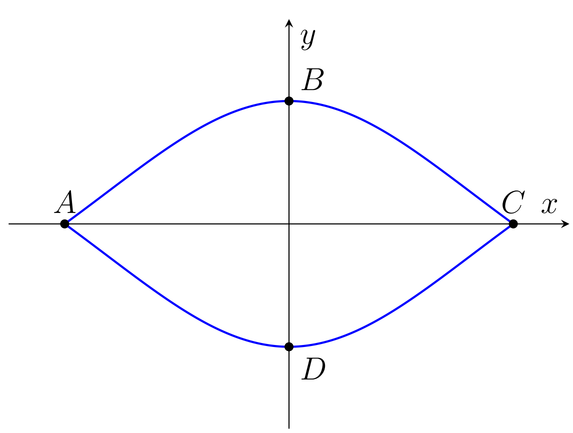
$b =$

$c =$

$d =$

Question 14

A plan of a park is shown in the diagram below. The boundary of the park is symmetrical about both axes and the curve ABC is modelled by part of the graph with equation $y = \dfrac{288}{x^2+32} - 6$ where x and y are measured in kilometres.

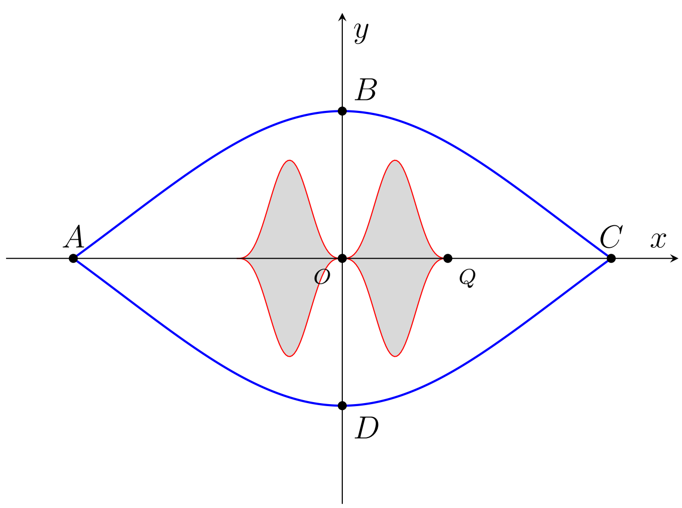


a) Find the distances OB and OC.

b) **i.** Using calculus find an anti-derivative of $y = \dfrac{288}{x^2 + 32} - 6$.

ii. Hence, find the area of the park according to this model stating your answer to two decimal places.

c) The shaded area on this diagram shows the position of two ornamental lakes in the park. The shaded area is symmetrical about both axes and the curve from O to Q is modelled by the part of the graph with equation $y = 2\sin^3(2x)$.



Using calculus, find the total area of the lakes according to this model.

Question 15

f is a function with rule $f(x) = \dfrac{x}{\sqrt{a^2 - x^2}}$, where a is a positive real number.

a) Find the maximal domain of f (in terms of a).

b) Show, using the quotient rule, that $f'(x) = \dfrac{a^2}{(a^2 - x^2)^{\frac{3}{2}}}$ and explain why there is no stationary point on the graph of f.

c) Find $f''(x)$ and find the coordinates of the point of inflection of the graph of f.

d) Find (in terms of a) the gradient of the graph of f at the point of inflection.

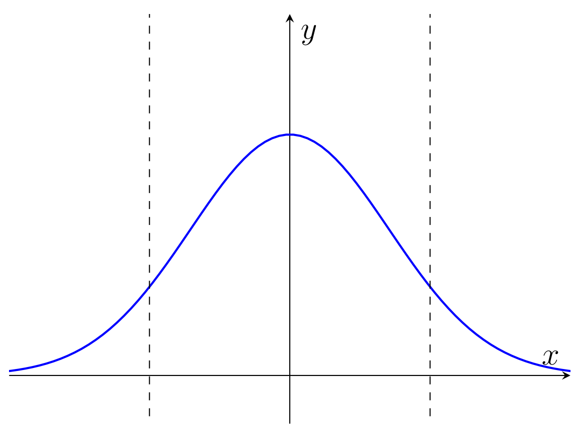
e) For the particular function f where $a = 2$, find the value of h such that the area of the region enclosed by the graph of f, the x-axis and the lines $x = -h$ and $x = h$ are equal to 1.

Question 16

a) Show, using a suitable substitution that $\int x^{n-1}e^{-x^n}\ dx = -\frac{1}{n}e^{-x^n} + c$, where n is a positive integer.

b) Hence, find the exact value of $\int_0^2 xe^{-x^2}\ dx$

c) The graph of $y = e^{-x^2}$ is shown below along with the graphs of $x = a$ and $x = -a$, where a is a positive constant.



i. Shade the region bounded by the curve, the lines $x = a$, $x = -a$ and $y = 0$.

ii. Write down an expression which gives the volume of the solid obtained by revolving the shaded region about the y-axis.

d) **i.** Show that $\dfrac{d}{dy}(y \log_e y) = 1 + \log_e y$

ii. Hence find $\displaystyle\int \log_e y \; dy$.

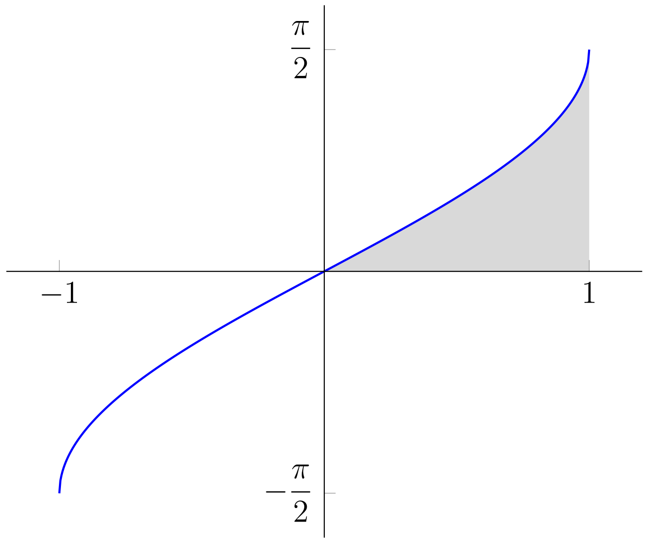
e) Find the volume of the solid of revolution defined by your integral in **c. ii.** in exact values.

f) What happens to the value of this volume as $a \to \infty$.

Question 17

a) Show that if $y = x\sin^{-1}(x) + \sqrt{1-x^2}$, then $\dfrac{dy}{dx} = \sin^{-1}(x)$.

b) Use your answer from above to evaluate the area of the shaded region, which is the area enclosed by the graph of $y = \sin^{-1}(2x)$, the x-axis and the line $x = \dfrac{1}{2}$.

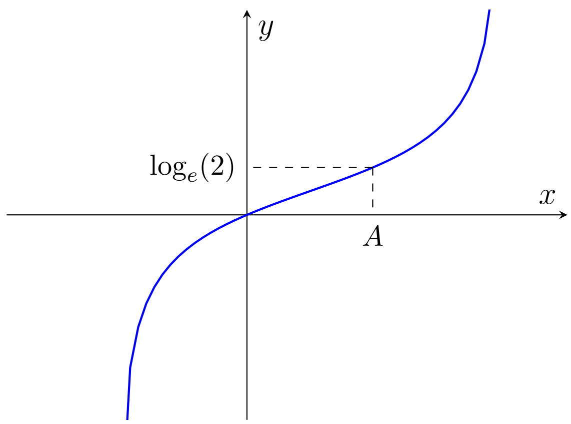


c) The shaded area can also be evaluated by first finding the area enclosed by the graph of $y = \sin^{-1}(2x)$, the y-axis and the line $y = \dfrac{\pi}{2}$.
Use this method to find the shaded area.

d) Find the volume of the solid of revolution formed when the shaded area is rotated around the y-axis.

Question 18

The graph of $f(x) = \log_e(1 + \tan(x))$ is shown below.



a) Show that $A = \dfrac{\pi}{4}$.

b) Show that $\tan\left(\dfrac{\pi}{4} - \theta\right) = \dfrac{1 - \tan(\theta)}{1 + \tan(\theta)}$

c) The region R is defined as the area bounded by $y = f(x)$, the x-axis and the line $x = A$. Let I represent the area of region R.

i. Express I as an integral.

ii. By using the substitution $u = \dfrac{\pi}{4} - x$, show that $2I = \displaystyle\int_0^{\frac{\pi}{4}} \log_e(2)\ du$.

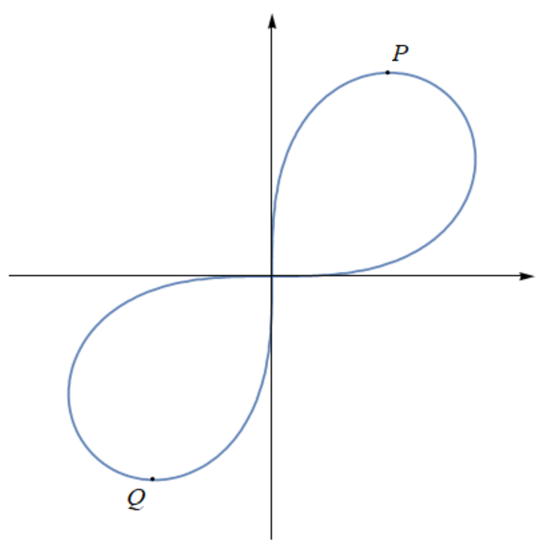
iii. Hence, find the exact value of the region R.

Question 19

The implicit function g is defined by the equation

$$(x^2 + y^2)^2 = 2xy$$

A sketch of the graph of g is shown below. On the sketch P and Q represent points on g where the tangent is horizontal.



a) Show that for the curve $\dfrac{dy}{dx} = \dfrac{2y - 4x(x^2 + y^2)}{4y(x^2 + y^2) - 2x}$.

b) Show that at the points P and Q, $x^2 + y^2 = \dfrac{y}{2x}$.

c) Hence, show that at points P and $Q, y = 8x^3$.

d) Find the coordinates of P and Q, expressing the y-values in the form $\left(\dfrac{a}{b}\right)^{\frac{3}{4}}$.

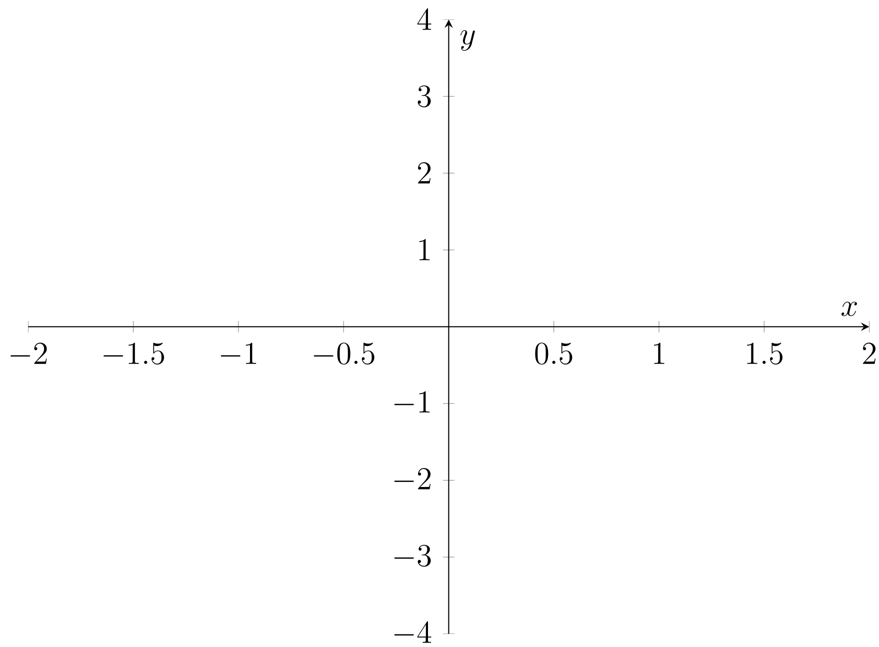
Case Study 1: Hyperbolic Functions

Question 1

In this starting point you will explore some properties of the hyperbolic functions $\cosh(x)$ and $\sinh(x)$, particularly with the circular functions.
Define, for any real number x:

$$\cosh(x) = \frac{1}{2}(e^x + e^{-x})$$
$$\sinh(x) = \frac{1}{2}(e^x - e^{-x})$$

Sketch the graph of the curves $y = \cosh(x)$ and $y = \sinh(x)$ on the same axes.
Give the range of each function.



Question 2

Considering the graph of $\sinh(x)$ and $\cosh(x)$, nominate a function $h(x)$ in terms of e^x such that $\sinh(x) > h(x) > \cosh(x)$ for $x \in \mathbb{R}$

Question 3

Show that

a) $\cosh^2(x) - \sinh^2(x) = 1$

b) $\cosh(2x) = 1 + 2\sinh^2(x)$

c) $\dfrac{d}{dx}(\cosh(x)) = \sinh(x)$

d) $\sinh(2x) = 2\sinh(x)\cosh(x)$

Question 4

a) By rewriting the equation as a quadratic in e^x, show that the exact value of x which satisfies the equation $4\sinh(x) = 3$ is $\log_e(2)$.

b) Explain why $\sinh(x)$ has an inverse function over the real numbers, while $\cosh(x)$ does not.

c) Consider the inverse if the $\sinh(x)$ function. State its domain and range.

d) Using the quadratic formula, show that $\text{arcsinh}(x) = \log_e(x + \sqrt{x^2 + 1})$ and hence verify your answer to part **a.** of this question.

Question 5

Prove the following identity:

$$(\cosh(x) + \sinh(x))^n = \cosh(nx) + \sinh(nx)$$

Question 6

Knowing that the hyperbolic tangent, $\tanh(x) = \dfrac{\sinh(x)}{\cosh(x)}$ and that the hyperbolic secant, $\mathrm{sech}(x) = \dfrac{1}{\cosh(x)}$, show that:

a) $\tanh(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

b) $\dfrac{d}{dx}(\tanh(x)) = \mathrm{sech}^2(x)$.

Question 7

Calculate:

a) $\displaystyle\int \frac{4}{\sqrt{1-16x^2}}\,dx$

b) $\displaystyle\int \frac{2x}{\sqrt{1-x^4}}\,dx$

c) $\displaystyle\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx$

d) From your answers above, propose a general solution to $\displaystyle\int \frac{f'(x)}{\sqrt{1-[f(x)]^2}}\,dx$.

Question 8

Now consider the integral $\int \sqrt{1-x^2}\, dx$.

a) Use the substitution $x = \sin(u)$ to solve the integral.

b) Next, try using the substitution involving the hyperbolic sine function $x = \sinh(u)$, to solve $\int \frac{1}{\sqrt{1+x^2}}\, dx$.

c) Hence, propose a general solution to integrals of the form $\int \frac{f'(x)}{\sqrt{1+[f(x)]^2}}\, dx$.

d) Hence, show that the value of $\int_0^{\frac{\pi}{2}} \frac{2\cos(x)}{\sqrt{1+\sin^2(x)}}\, dx$ is $2\log_e(1+\sqrt{2})$

e) From your work above, and by substituting $x = \frac{1}{u'}$ find an anti-derivative of the function:

$$f : (0, \infty) \to \mathbb{R}, f(x) = \frac{1}{x\sqrt{1+x^2}}$$

Question 9

Verify that if P has position vector $\underset{\sim}{r} = \cos(\theta)\underset{\sim}{i} + \sin(\theta)\underset{\sim}{j}$ where θ is any real number, then P is the unit circle.

Question 10

For points of the unit hyperbola, an equivalent set of parametric equations exists, namely

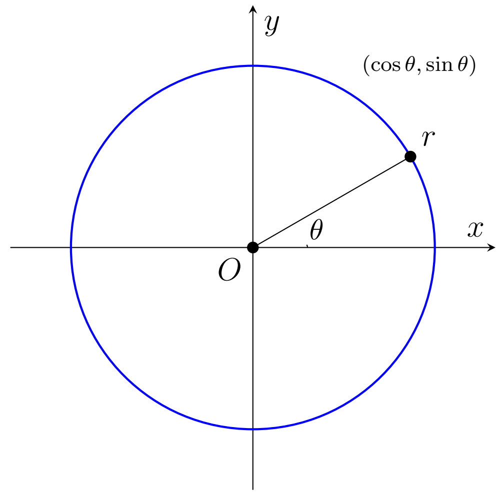
$$x = \cosh(t)$$

$$y = \sinh(t)$$

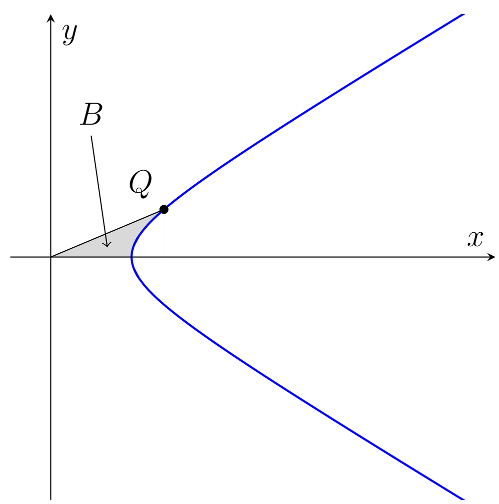
Find the Cartesian equation of the locus of a point Q, with position vector $\underset{\sim}{r} = \cosh(t)\underset{\sim}{i} + \sinh(t)\underset{\sim}{j}$

Question 11

a) The variable θ in $\underset{\sim}{r} = \cos(\theta)\underset{\sim}{i} + \sin(\theta)\underset{\sim}{j}$ can be interpreted geometrically as an angle, which is measured in radians.
Show that $\theta = 2A$, for $0 < \theta < 2\pi$, where A is the magnitude of the area of the shaded region in the diagram below.



The variable t in $\underset{\sim}{r} = \cosh(t)\underset{\sim}{i} + \sinh(t)\underset{\sim}{j}$ cannot be similarly interpreted as an angle, but we can say that t is proportional to the area B shown in the diagram below.

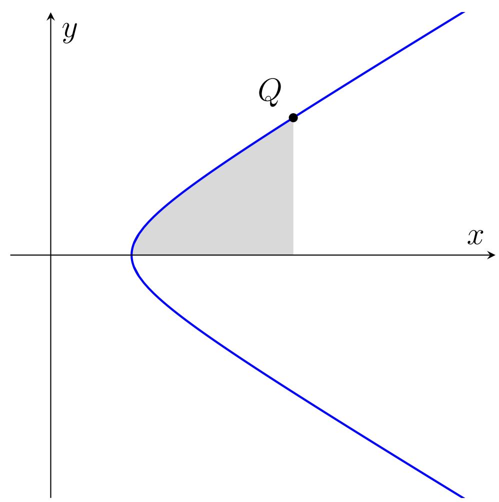


b) Express $\underset{\sim}{r} = \cosh(t)\underset{\sim}{i} + \sinh(t)\underset{\sim}{j}$ in Cartesian form.

c) To determine the relationship between t and B for $t \geq 0$, we first express the area C in the diagram below as a definite integral.

Given $C = \displaystyle\int_1^x \sqrt{x^2 - 1}\ dx$,

show that $C = \dfrac{1}{4}\sinh(2t) - \dfrac{1}{2}t$.



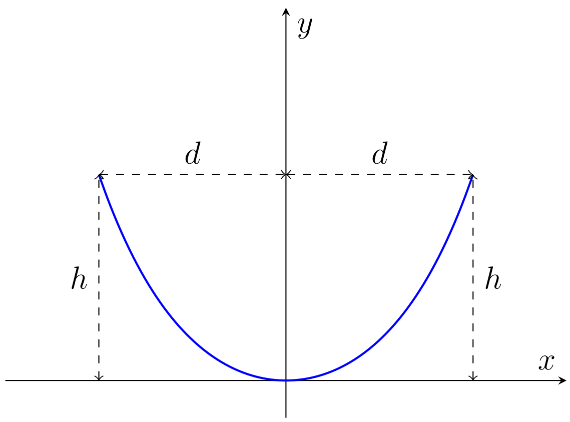
d) Hence, show that $t = 2B$.

Question 12

A flexible chain of length l hangs loosely between two poles of equal height a distance $2d$ metres apart, so that it sags a distance h metres in the centre.

The curve formed by the chain is called a catenary. Using a coordinate system with the lowest point at the origin, the catenary can be describe by the equation:

$$y = \frac{1}{m}\cosh(mx) - \frac{1}{m}$$



a) **i.** Show that $\dfrac{dy}{dx} = \sinh(mx)$.

ii. Hence show that $\left(\dfrac{dy}{dx}\right)^2 = \dfrac{e^{2mx} + e^{-2mx} - 2}{4}$.

b) Use the arc length formula to show that the length of the chain is given by

$$\frac{e^{md} - e^{-md}}{m}$$

c) In a particular catenary, where $m = \dfrac{1}{32}$, it is required that the height, h, is 20 metres. Find, correct to two decimal places

i. The distance between poles.

ii. The length of the cable

d) If the length of the cable is 100 metres and the distance between the poles is 80 metres, find the value of m, correct to two decimal places.