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Specialist Maths Units 3/4

Complex Numbers

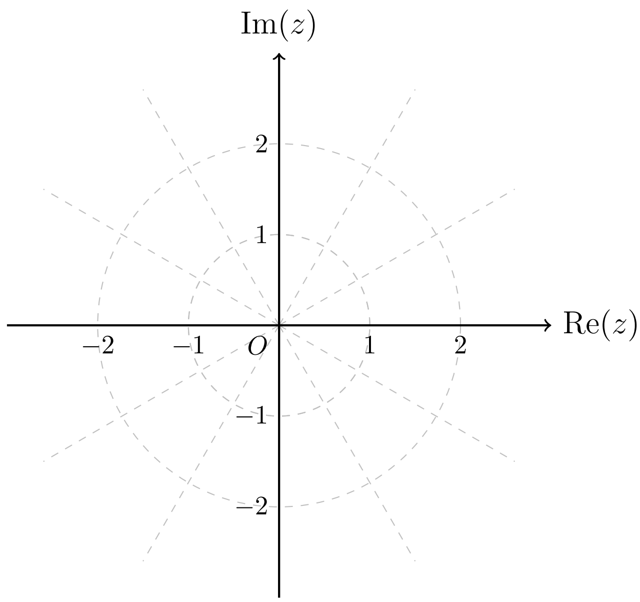
Practice Questions

Short Answer Questions

Question 1

a) Express $z_1 = -\frac{\sqrt{3}}{2} + \frac{i}{2}$ in polar form.

b) Plot and label z_1 on the Argand diagram below:



c) Also label the following points on the same diagram:

i. $z_2 = 2iz_1$

ii. $z_3 = \overline{(z_1)^3}$ (the conjugate of the cube of z_1)

Question 2

a) **i.** Let $z_1 = \sqrt{2} + \sqrt{2}i$. Express z_1 in polar form.

ii. Find $\overline{z_1}$ in polar form.

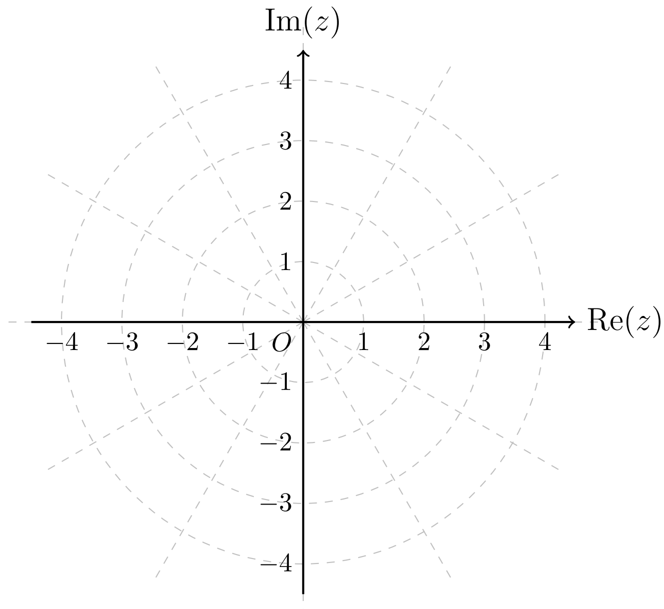
iii. Let $z_1 = 2\text{cis}\left(\frac{3\pi}{4}\right)$. Show that $\frac{z_1}{z_2} = -i$.

If z_1, z_2 z_3 are the three corners of an equilateral triangle on an Argand diagram, find z_3.

Question 3

a) Express $\{z : z\overline{z} = 1\}$ in Cartesian form. Describe the shape of this region.

b) Shade the region $\{z : z\overline{z} \leq 1\} \cap \{z : \text{Arg}(z) > \frac{2\pi}{3}\}$ on the Argand diagram below.

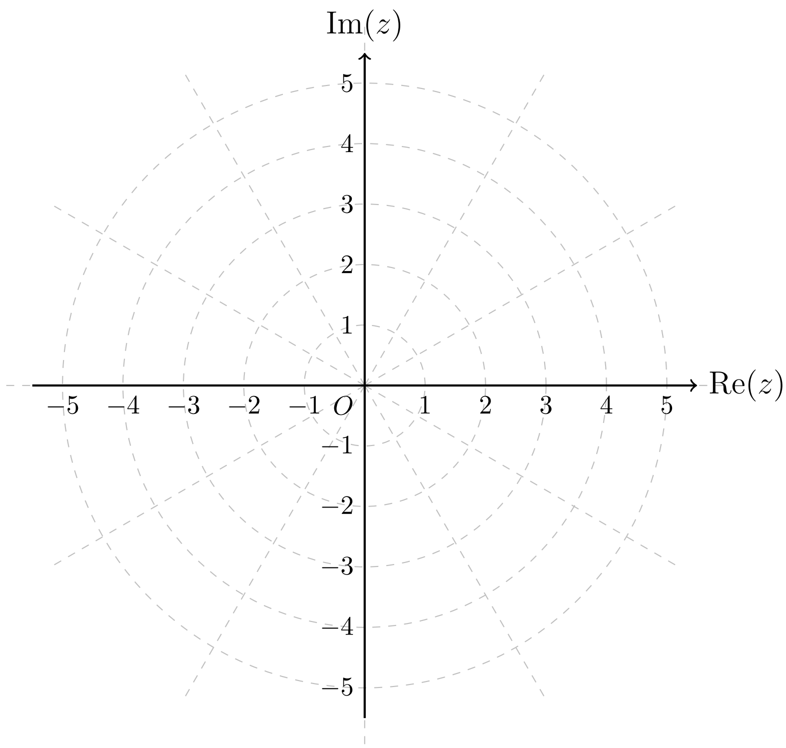


Question 4

Find in polar form all numbers $z \in \mathbb{C}$ such that $z^3 + 4 - 4\sqrt{3}i = 0$

Question 5

On the Argand diagram below shade the region $R = T \cap S$, where
$T = \{z : |z - 2 + 2i| < 2\}$ and $S = \{z : -\frac{\pi}{2} \leq \text{Arg}(z) \leq -\frac{\pi}{4}\}$.



Question 6

Consider the graph with rule $(z + 2 - i)(\overline{z} + 2 + i) = 4$ where $z \in \mathbb{C}$.
Write this rule in Cartesian form.

Question 7

Solve for z:

a) $\dfrac{1}{\sqrt{z}} = a$ where $a = \sqrt{3} + i$. Give your answer in terms of a.

b) $\dfrac{1}{\sqrt[3]{z}} = \sqrt{3} + i$. Give your answer in the form $\dfrac{1}{\sqrt[a]{b}}(\text{cis}(\theta))$

c) $z^2 = 9i$. Give your answer in the form $r(\text{cis}(\theta))$

Question 8

Let $z = x + iy$ be a complex number where x and y are real.

a) $\dfrac{1}{1-z}$ in the form $a + bi$, where a and b are real quantites that depend on x and y.

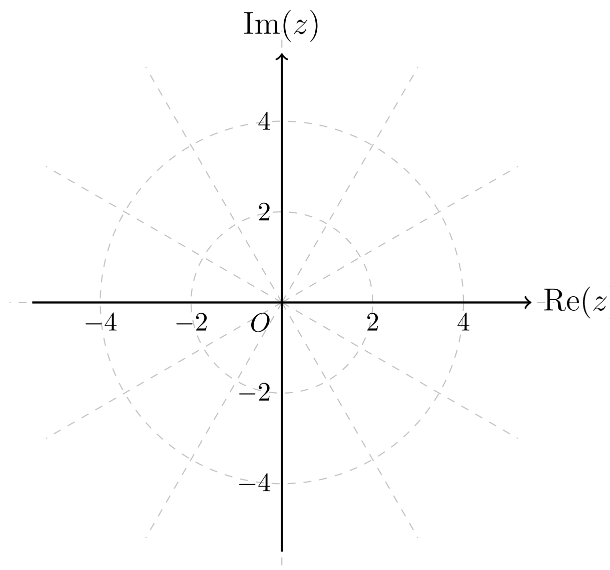
b) Hence, show that the relation $S = \left\{ z : \text{Im}\left(\dfrac{1}{1-z}\right) < 1 \right\}$ represents the exterior of a circle.
State the centre and radius of this circle.

c) Express S in the form $|z + m + ni| > p$, where $m, n, p \in \mathbb{R}$

Question 9

The locus T defined by $T = \{z : |z + 2 - 2i| = 4\}$ while the locus S is defined by $S = \{z : \text{Arg}(z + 2 + 4i) = \theta, 0 < \theta < \frac{\pi}{2}\}$ such that S and T have only one point of intersection.

a) On the axes below sketch the regions defined by S and T. State the centre of T and the starting point of S.



b) Show that $\sin(\theta) = \dfrac{\sqrt{5}}{3}$

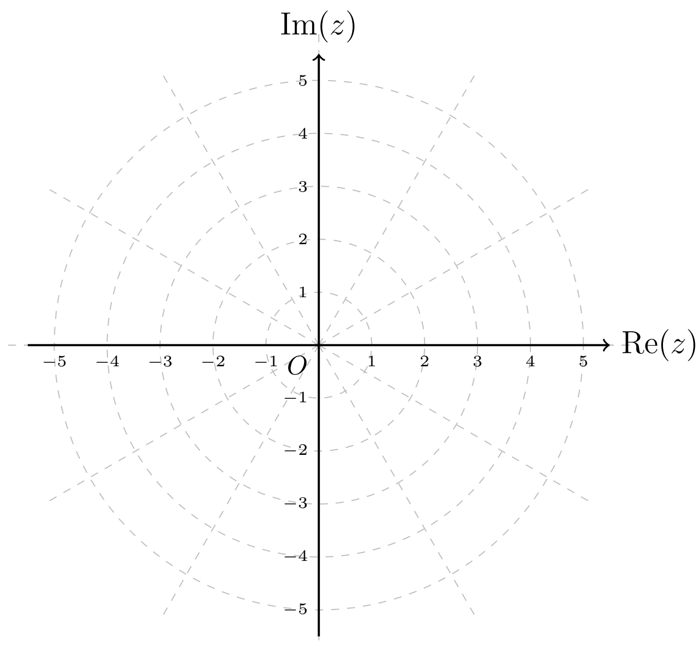
c) The complex number w represents the intersection point of S and T. Find w in Cartesian form.

Question 10

Let $v = 2\sqrt{3} - 2i$ and $w = \sqrt{3} + i$.

a) Find $\text{Arg}(v^4)$

b) On the Argand diagram on the right, label the two complex numbers w and $2\overline{w}i$



c) Algebraically find $\text{Im}(v \cdot w + \frac{v}{w})$

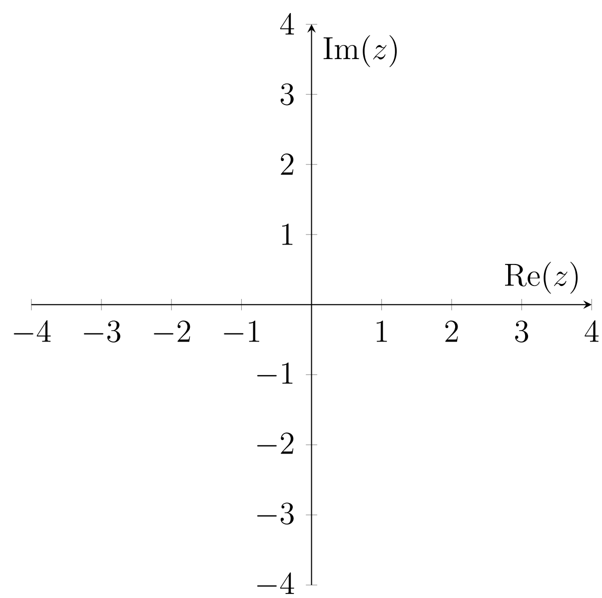
Question 11

Consider the graph with the rule $|z + 2 - i| = 2$ where $z \in \mathbb{C}$.

a) Write this rule in Cartesian form.

b) Find the points of intersection of the graph with rules $|z + 2 - i| = 2$ and $\text{Arg}(z) = \pi$ in Cartesian form.

c) Sketch and shade the region $\{z : |z + 2 - i| \leq 2\} \cap \{z : -\pi < \text{Arg}(z) \leq 0\}, z \in \mathbb{C}$ on the Argand diagram on the right.



d) Calculate the area of the shaded region found in part **c**.

Question 12

$z_1 = 2\sqrt{2}\text{cis}\left(\frac{\pi}{4}\right)$ and $2\text{cis}\left(\frac{2\pi}{3}\right)$

a) Find

i. $z_1 z_2$

ii. $\frac{z_1}{z_2}$

b) z_1 is rotated $\frac{\pi}{2}$ about the origin in an anticlockwise direction to a new point z_3.
Find z_3

c) Express z_2 in Cartesian form.

d) In Cartesian form $z_1 = 2 + 2i$. Given that z_1 is a solution of the equation $z^4 - 2z^3 + 3z^2 + 4z + 24 = 0$ find the other three solutions.

Question 13

For $w = \text{cis}(\theta)$

a) Find w^2 and w^{-2} in polar form.

b) Let $z = w + \dfrac{1}{w}$

i. Find z^2 in terms of θ and state why z^2 is real.

ii. Find the maximum and minimum values of z^2.

Question 14

Represent the following subsets of $\mathbb{C}$ on an Argand diagram, include its Cartesian equation.

$$|z + 5i| - |z - i| = 4$$

Question 15

Given $z_1\sqrt{3}+i$ and $z_2=-\sqrt{2}-\sqrt{2}i$

a) Express $\dfrac{z_1}{z_2}$ in Cartesian form.

b) Express z_1 and z_2 in polar form.

c) Express $\dfrac{z_1}{z_2}$ in polar form.

d) Hence show that $\sin\left(\dfrac{11\pi}{12}\right)=\dfrac{\sqrt{6}-\sqrt{2}}{4}$ and $\cos\left(\dfrac{11\pi}{12}\right)=\dfrac{-\sqrt{6}-\sqrt{2}}{4}$

Question 16

Given that $x^2 + ix + 6 = 5x + 2i$,

a) Find the possible value(s) of x if x is a real number.

b) Find the possible value(s) of x if x is a complex number.

Question 17

a) Show that the cubic equation

$$z^3 - 3\sqrt{3}iz^2 - 9z + 3\sqrt{3}i = -4\sqrt{3} - 4i$$

b) Can be written in the form of

$$(z - \sqrt{3}i)^3 = -4\sqrt{3} - 4i$$

c) Hence, find in exact Cartesian form, all the solutions of the equation.

$$z^3 - 3\sqrt{3}iz^2 - 9z + (3\sqrt{3} + 4)i + 4\sqrt{3} = 0$$

Question 18

Let $z_2 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$

a) Express z_2 in polar form.

b) Let $z_1 = a + bi$ and $z_1^2 = z_2$.

Using Cartesian methods, find a and b in the form $\frac{\sqrt{p+\sqrt{q}}}{r}$ and $\frac{\sqrt{p-\sqrt{q}}}{r}$ respectively where $p, q, r \in \mathbb{Z}$.

c) Hence, find expressions for $\cos\left(\frac{\pi}{8}\right)$ and $\sin\left(\frac{\pi}{8}\right)$

d) Hence, express $\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)^{\frac{17}{2}}$ in Cartesian form.

Question 19

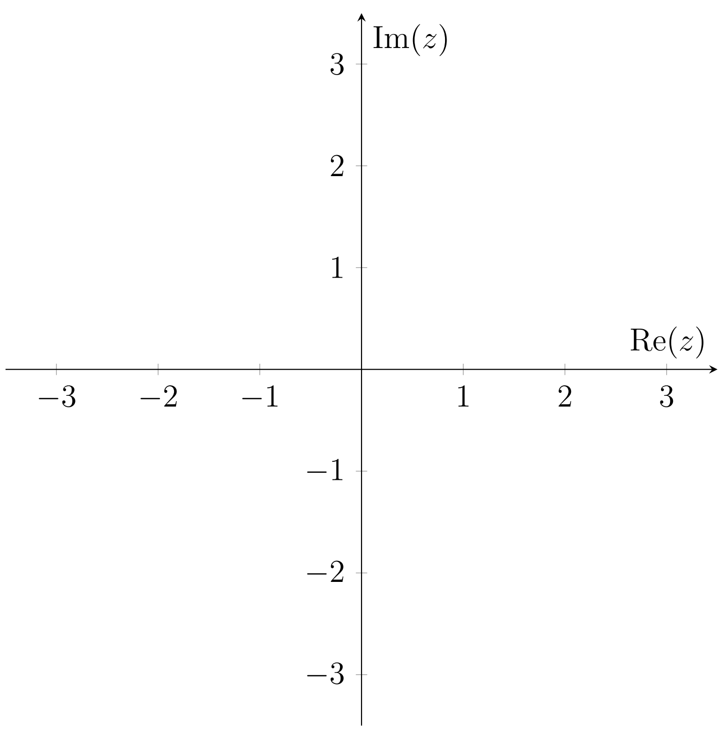
Consider the equation $z^4 - 2z^3 + 7z^2 - 4z + 10 = 0, z \in \mathbb{C}$

a) Use algebra of find real values of a for which ai is a solution to this equation.

b) Hence find all complex solutions to this equation.

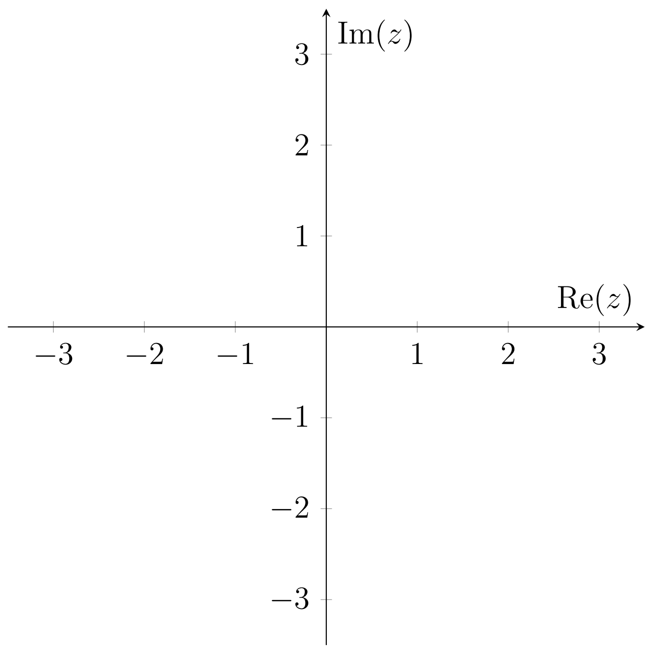
Question 20

a) Sketch the ponts $S = \{z : (z - \sqrt{2})(\overline{z} - \sqrt{2}) = 2, z \in \mathbb{C}\}$ on the Argand diagram below.



b) Show that, with the exception of a single value of z which should be stated, the points in S satisfy $\text{Arg}(z - \sqrt{2}) = 2\text{Arg}(z)$.

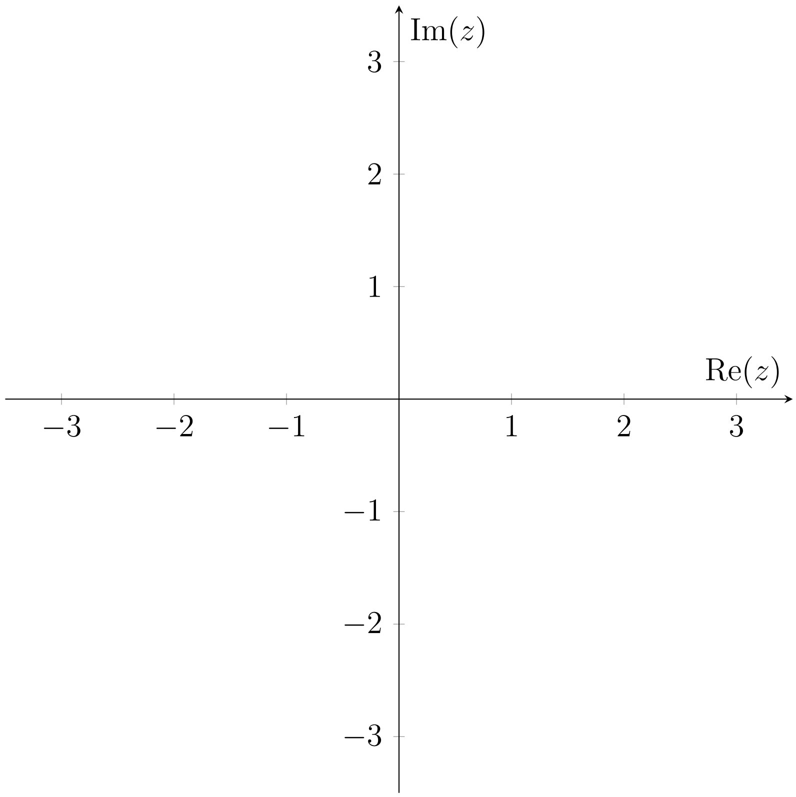
c) Sketch the set of points $T = \{z : \text{Arg}(z - \sqrt{2}) = 2\text{Arg}(z), z \in \mathbb{C}\}$ on the Argand diagram below.



d) Find in exact polar form all points in T that satisfy $|z| = |z - 3\sqrt{2}|$.

Question 21

Sketch the set of points $\{z : \text{Arg}(zi - 1) = \dfrac{\pi}{3}, z \in \mathbb{C}\}$ on the Argand diagram below.



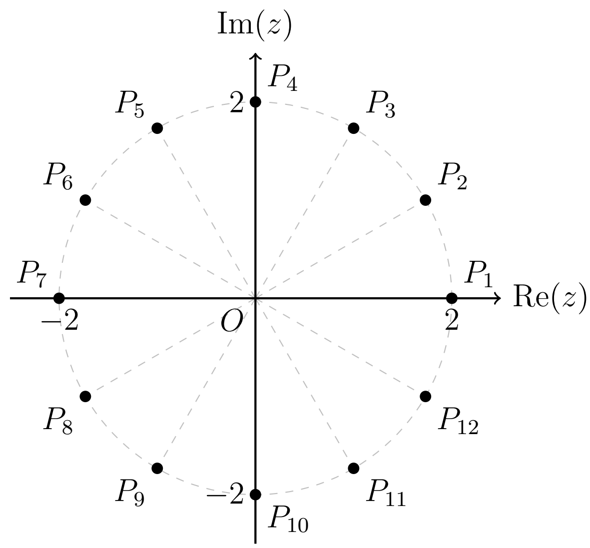
Question 22

Factorise $z^2 + (2 - 2i)z = -1 - 2i$ over $\mathbb{C}$

Multiple Choice Questions

Question 1

Which points on the diagram below represents the complex number z such that $z^3 = 8i$?



A. P_{10} only

B. P_3, P_5 and P_{10}

C. P_4, P_8 and P_{12}

D. P_2, P_6 and P_{10}

E. P_1 only

Question 2

For any complex number z, the location on an Argand diagram of the complex plane $u = i^3\overline{z}$ can be found by:

A. rotating z through $\frac{3\pi}{2}$ in an anticlockwise direction about the origin.

B. reflecting z about the real axis and the reflecting it about the imaginary axis.

C. reflecting z about the imaginary axis and then rotating anticlockwise through $\frac{\pi}{2}$ about the origin.

D. reflecting z about the real axis and then rotating anticlockwise through $\frac{\pi}{2}$ about the origin.

E. rotating z through $\frac{3\pi}{2}$ in an clockwise direction about the origin.

Question 3

The set of points in the complex plane defined by $|z| = |z + 3i|$ is represented by:

A. The circle with centre (3,0) and radius 3.

B. The circle with centre (−3,0) and radius 3.

C. The line $\text{Im}(z) = -\frac{3}{2}$

D. The line $\text{Im}(z) = \frac{3}{2}$

E. The point $z = -\frac{3i}{2}$

Question 4

Which of the following does not represent a circle in the complex plane?

A. $|z - 1| = 9$

B. $|z - 3| = 4$

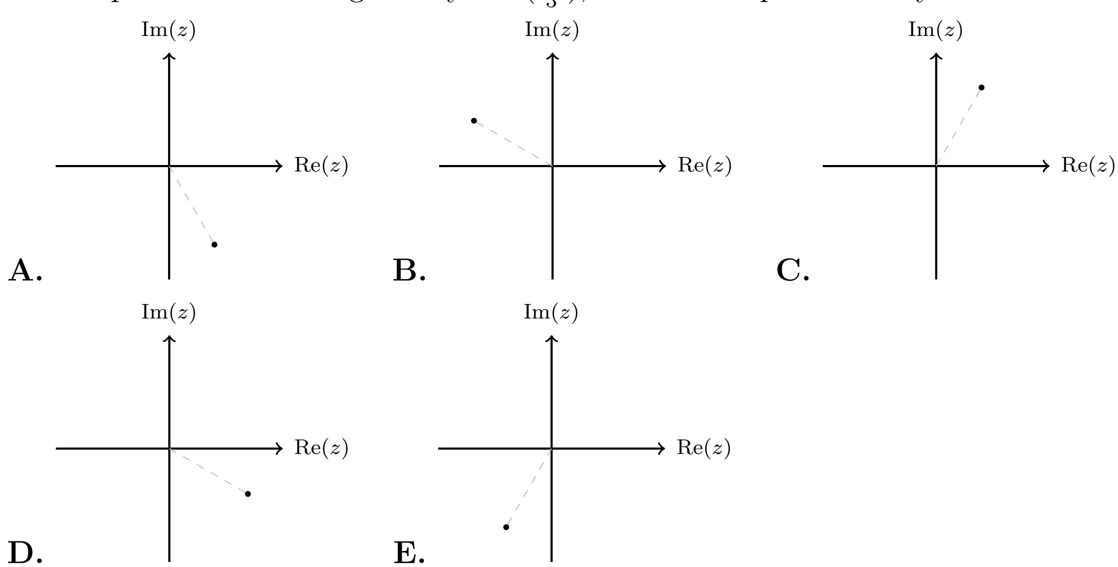
C. $(z - 3)(\overline{z} - 3) = 4$

D. $|z - 3| = |z|$

E. $|z + 1| = 2|z|$

Question 5

The complex number z is given by $r\text{cis}(\frac{2\pi}{3})$, $i\overline{z}$ is best represented by:



Question 6

If $z = a + bi$, where $a, b \in \mathbb{R}$, then $\dfrac{|iz|^2}{\overline{z}}$ simplifies to:

A. i

B. 1

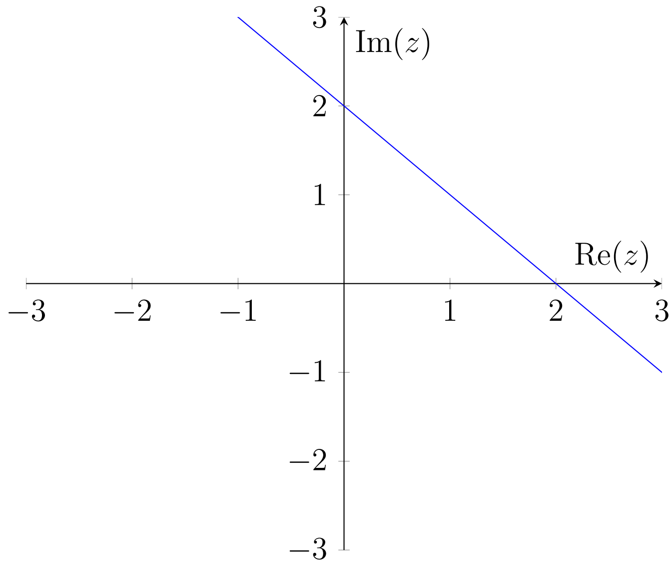
C. z

D. $-z$

E. iz

Question 7

The equation of the line shown in the diagram below is:



A. $\{z : \text{Arg}(z) = \dfrac{3\pi}{4}\}$

B. $\{z : \text{Arg}(z) = -\dfrac{\pi}{4}\}$

C. $\{z : \text{Arg}(z) = -\dfrac{\pi}{4} + 2\}$

D. $\{z : \text{Re}(z) + \text{Im}(z) = 0\}$

E. $\{z : \text{Re}(z) + \text{Im}(z) = 2\}$

Question 8

$(a \text{ cis}(\frac{\pi}{3}))^3 \cdot (b \text{ cis}(\frac{\pi}{6}))^2$ simplifies to:

A. $a\ b \text{ cis}(\frac{\pi}{2})$

B. $a^3\ b^2 \text{ cis}(\frac{\pi}{2})$

C. $a\ b \text{ cis}(\frac{4\pi}{3})$

D. $a^3\ b^2 \text{ cis}(-\frac{2\pi}{3})$

E. $a\ b \text{ cis}(-\frac{2\pi}{3})$

Question 9

If $(x + yi)^2 = 18i$ for real values of x and y, then:

A. $x = 3$ and $y = 3$

B. $x = -3$ and $y = 3$

C. $x = 3$ and $y = -3$

D. $x = 3, y = -3$ and $x = -3, y = 3$

E. $x = 3, y = 3$ and $x = -3, y = -3$

Question 10

One of the linear factors of $x^2 + ax + 10$, where a is a real number is $(x - 3 + i)$. The value of a is:

A. 3

B. $-3 - \dfrac{10}{3}$

C. -3

D. -6

E. 6

Question 11

Let $u = \sqrt{3} - i$ and $v = i - 1$. Then $|uv|$ is equal to:

A. $\sqrt{3}$

B. $2\sqrt{2}$

C. 2

D. 0

E. 1

Question 12

The subset of the complex plane defined by the relation $|z + 5| - |z| = 0$ is:

A. A circle

B. An ellipse

C. A straight line

D. A null set

E. A hyperbola

Question 13

The Cartesian equation of the locus defined by $\{z : \text{Arg}(z - 1) < -\frac{3\pi}{4}\}$ is:

A. $y > x + 1, x < 1$

B. $y < -x + 1, x < -1$

C. $y < x + 1, x < -1$

D. $y > x - 1, y < 0$

E. $y < -x - 1, x > 1$

Question 14

Let $S = \{z : |z - 3 + 2i| \leq 1\}$. If $z \in S$, then the maximum value of $|z|$ is:

A. $\sqrt{13} - 1$

B. $\sqrt{13}$

C. $\sqrt{13} + 1$

D. 5

E. 7

Question 15

If a polynomial with real coefficients has solutions $z = 1 - ai$ and $z = 2$ over the set of complex numbers numbers, where a is real, then its quadratic factor must be.

A. $z^2 + 4$

B. $z^2 - 2aiz + 1$

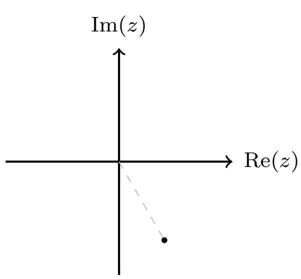
C. $z^2 - 2z + 1 + a^2$

D. $z^2 - 4$

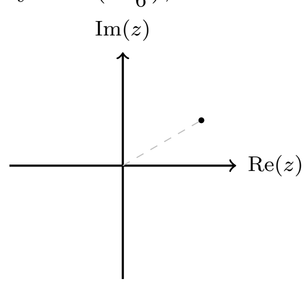
E. $z^2 + 2z + 1 - a^2$

Question 16

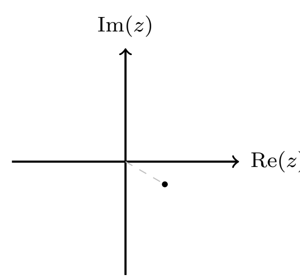
The complex number z is given by $r\text{cis}(-\frac{\pi}{6})$, where $r > 1$, $\frac{1}{z}$ is best represented by:



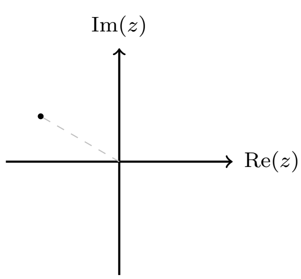
A.



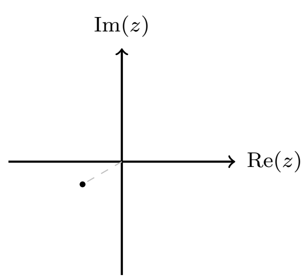
B.



C.



D.



E.

Question 17

If $z = \pi \text{cis}(3)$, then $\text{Arg}(z^2)$ is equal to:

A. 9

B. 6

C. 0

D. $6 - 2\pi$

E. $6 - \pi$

Question 18

Let $z = \dfrac{1}{\sin\theta} + \dfrac{1}{\cos\theta} i$ and $\theta \in [\dfrac{\pi}{2}, \pi]$. Then in polar form, z is equal to:

A. $\dfrac{1}{\sin\theta\cos\theta}\ \text{cis}(\theta)$

B. $\dfrac{2}{\sin 2\theta}\ \text{cis}(\theta - 2\pi)$

C. $\dfrac{-2}{\sin 2\theta}\ \text{cis}(\theta - \pi)$

D. $\dfrac{2}{\sin 2\theta}\ \text{cis}(2\pi - \theta)$

E. $\dfrac{-2}{\sin 2\theta}\ \text{cis}(-\theta)$

Question 19

Let $S = \{z : |z| \leq 3\} \cap \{z : |z - 3 - 3i| \leq 3\}$. Which of the following is correct ?

A. $\text{Arg}(z) \geq \dfrac{\pi}{2}$

B. $3\sqrt{2} - 3 \leq |z| \leq 3$

C. $\text{Arg}(z) = \dfrac{\pi}{4}$

D. $\sqrt{3} - 1 \leq |z| \leq 3$

E. $\text{Re}(z) - \text{Im}(z) = 3$

Question 20

If $|z| - z = 1 + 2i$, the:

A. $\text{Re}(z) + \text{Im}(z) = \dfrac{1}{2}$

B. $|z| = \sqrt{5}$

C. $\text{Arg}(z) = \tan^{-1}(2)$

D. $\text{Re}(z) < \text{Im}(z)$

E. $|z| = \dfrac{5}{2}$

Question 21

Given the complex number $z = a\text{cis}(\frac{\pi}{b})$ where $a, b \in \mathbb{R}\backslash\{0\}$. Then $\frac{1}{z}$ is equal to ?

A. $z = a\text{cis}(\frac{\pi}{b})$

B. $z = a\text{cis}(-\frac{\pi}{b})$

C. $z = \frac{1}{a}\text{cis}(\frac{\pi}{b})$

D. $z = \frac{1}{a}\text{cis}(-\frac{\pi}{b})$

E. $z = \frac{1}{a}\text{cis}(\frac{b}{\pi})$

Extended Response Questions

Question 1

Let $u = -\sqrt{3} - i$ and $P(z) = z^3 + mz^2 + nz - 12$, where $m, n \in \mathbb{R}$

a) Express $\overline{u}$ in the form $r\text{cis}(\theta)$

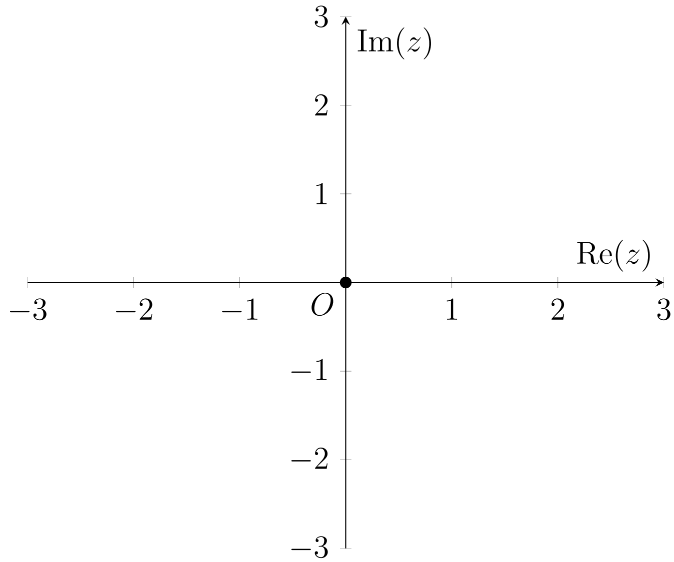
b) Show that $u + \overline{u} = 4\cos\left(\frac{5\pi}{6}\right)$

c) Express $(z - u)(z - \overline{u})$ in the form $z^2 + bz + c$, where $b, c \in \mathbb{R}$

d) If $u, \overline{u}$ and a are the roots of the equation $P(z) = 0$, find the value of the other root, a.

e) Hence state the exact value of m.

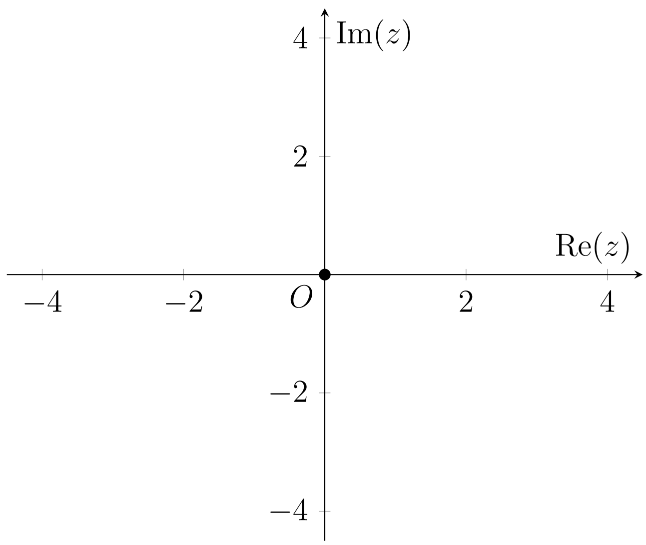
f) On the Argand plane below, plot the points $u, \overline{u}$ and a corresponding to the three roots of $P(z) = 0$, and describe the shape formed by connecting these three roots.



Question 2

Let $A = \{z : \text{Arg}(z+1) = \frac{\pi}{4}\}$ and $B = \{z : |z+1| = 2\}$

a) Sketch the complex regions A and B on the Argand diagram below:



b) Write down the Cartesian equations of the regions defined by A and B.

c) Show that $z_0 = (\sqrt{2} - 1) + \sqrt{2}i$ is the point of intersection of the regions A and B.

d) **i.** Find $(z_0 + 1)^6$ in Cartesian form.

ii. Find both values of $\sqrt{z_0 + 1}$ in polar form.

Question 3

a) **i.** Express the complex number $1+i$ in the form $r\text{cis}(\theta)$

ii. Hence, show that $(1+i)^{11} = -32+32i$.

b) **i.** Show that $z = 3+2i$ is a solution of the cubic equation
$z^3 - 8z^2 + 25z - 26 = 0, z \in \mathbb{C}$

ii. Hence state another non-real solution of the cubic equation
$z^3 - 8z^2 + 25z - 26 = 0$.

iii. Hence find a quadratic factor (with real coefficients) of $z^3 - 8z^2 + 25z - 26 = 0$.

iv. Hence fully factorise $z^3 - 8z^2 + 25z - 26$ over $\mathbb{C}$.

c) **i.** Expand $(\cos(\theta) + i\sin(\theta))^3$ without using **de Moivre's theorem**.

ii. By using **de Moivre's theorem**, show that $\cos(3\theta) = 4\cos^3(\theta) - 3\cos(\theta)$.

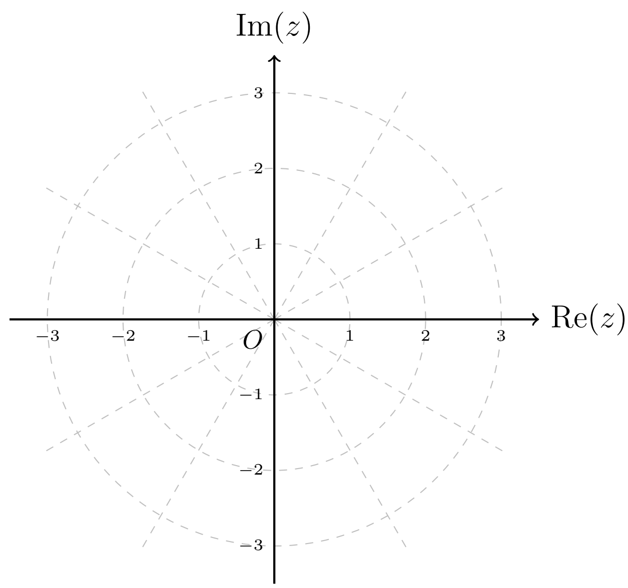
iii. Similarly, express $\sin(3\theta)$ in terms of $\sin(\theta)$ only.

Question 4

Let $z_1 = \sqrt{3} + i$.

a) Express z_1 in polar form.

b) Plot the roots of the equation $z^3 = 8i$ on the Argand diagram below, labelling each one clearly, expressing them in polar form.

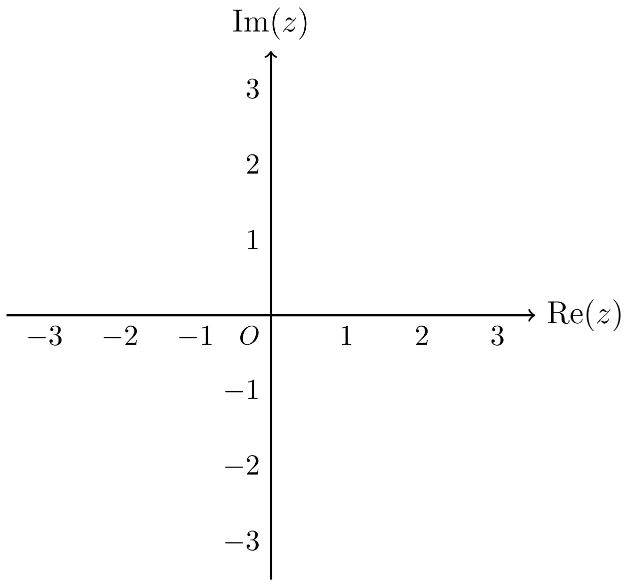


Let $z = x + yi$ and $z_1 = \sqrt{3} + i$ (from part a.)

c) Show that the Cartesian equation for the equation $z\overline{z} + |z_1| \cdot \text{Re}(i^2 z) - 2\text{Im}(z) = -1$ is given by the Cartesian equation given by $(x-1)^2 + (y-1)^2 = 1$

Let $S = \{z : z\overline{z} + |z_1| \cdot \mathrm{Re}(i^2 z) - 2\mathrm{Im}(z) \leq -1\} \cap \left\{0 \leq \mathrm{Arg}(z) \leq \frac{\pi}{4}\right\}$

d) Sketch S on the Argand diagram below.



e) If $z_2 \in S$. Find the maximum and minimum values of $|z_2|$. Express your values in exact form.

Question 5

Let $z_1 = 3 - 4i$ and $z_2 = 2 - i$

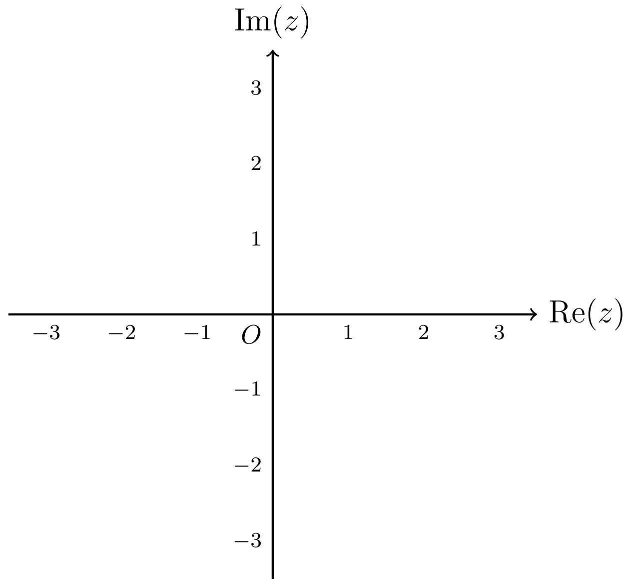
a) **i.** Express $\frac{1}{z_1}$ in Cartesian form.

ii. Find the values of $\text{Arg}(z_1)$ to the nearest minute.

iii. Verify that $\text{Im}(z_1) + \overline{z_2} + z_1 z_2 = -10i$

iv. Find algebraically the square root(s) of z_1 and express them in Cartesian form.

b) **i.** On the complex plane below sketch S where $S = \{z : |z - 2 + i| = |z + 2 - i|\}$.



ii. Find algebraically the Cartesian equation of S.

iii. Given that $z_4 = 3\text{cis}(\frac{\pi}{3})$, state with reasons whether or not $z_4 \in S$.

iv. Write down the subset of the complex plane that is equidistant from the points z_1 and z_4.

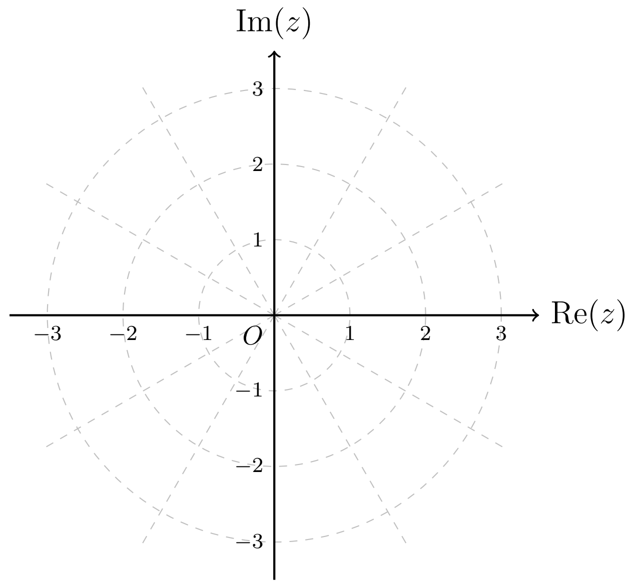
Question 6

a) Let $u = 2 + 2i$ and $v = 1 + \sqrt{3}i$. Express$u \cdot v$ in polar form.

b) If $\theta = \text{Arg}\left(\frac{u}{v}\right)$ show that $\theta = -\frac{\pi}{12}$

c) Hence, or otherwise, find an expression for $\left(\frac{u}{v}\right)^2$ in exact polar form.

d) Plot, and clearly label $\dfrac{u}{v}$ and $\left(\dfrac{u}{v}\right)^2$ on the Argand diagram.



e) Hence shade in, on the Argand diagram above, the region represented by:

$$\left\{z : \text{Arg}(z) \leq \text{Arg}\left(\frac{u}{v}\right)\right\} \cap \left\{\text{Arg}(z) > \text{Arg}\left(\frac{u}{v}\right)^2\right\} \cap \{z : |z| \leq 2\}$$

Question 7

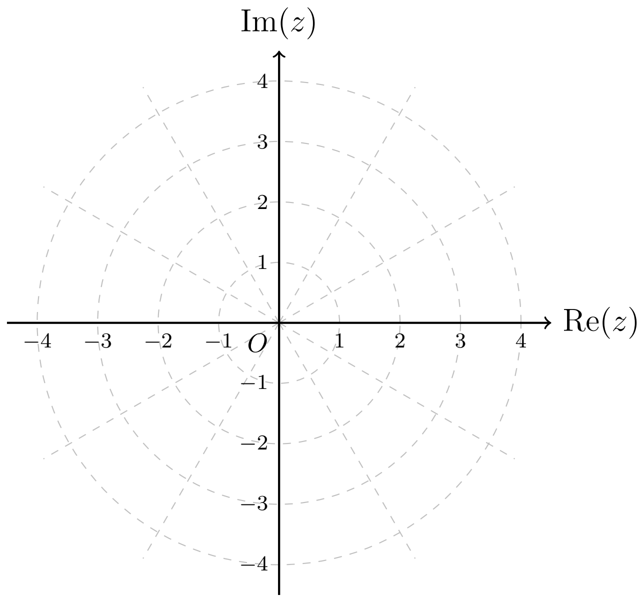
a) Express $-4\sqrt{2}+4\sqrt{2}i$ in exact polar form.

b) Show that one of the cube roots of $-4\sqrt{2}+4\sqrt{2}i$ is $u = 2\text{cis}\left(\frac{\pi}{4}\right)$

c) Find the remaining two cube roots of $-4\sqrt{2}+4\sqrt{2}i$ in exact polar form.

d) Express $u = 2\text{cis}\left(\frac{\pi}{4}\right)$ in exact Cartesian form.

e) Plot the three cube roots of $-4\sqrt{2}+4\sqrt{2}i$ on the Argand diagram below.



f) Show that the cubic equation $z^3-3\sqrt{2}iz^2-6z=-4\sqrt{2}+2\sqrt{2}i$ can be expressed in the form $(z-w)^3=-4\sqrt{2}+4\sqrt{2}i$ where w is a complex number.

g) Hence find, in exact Cartesian form, one root of the equation:

$$z^3-3\sqrt{2}iz^2-6z=-4\sqrt{2}+2\sqrt{2}i$$

Question 8

a) Find the Cartesian equation of the set of points defined by:

i. $iz - i\overline{z} = -9$, where $z \in \mathbb{C}$

ii. $|z - 3 - 4i| = 1$, where $z \in \mathbb{C}$

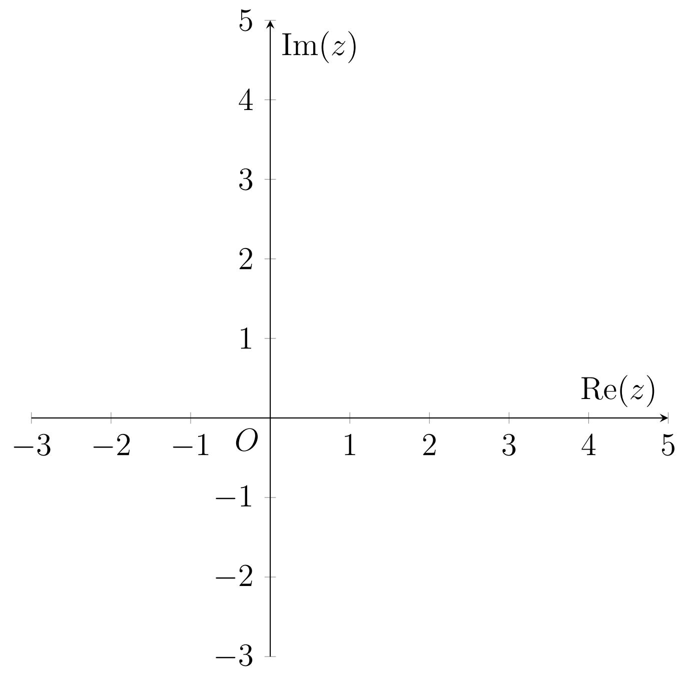
iii. $\text{Arg}(z - 1 + i) = \frac{2\pi}{3}$, where $z \in \mathbb{C}$

Consider the regions:
$S = \{z : |z - 3 - 4i| \leq 1, z \in \mathbb{C}\}$ and
$T = \{z : |z| \leq 2, z \in \mathbb{C}\} \cap \left\{z : \text{Arg}(z - 1 + i) \leq \frac{2\pi}{3}\right\}$

b) On the Argand diagram below, shade the regions

i. S

ii. T



Let $z_1 \in T$ and $z_2 \in S$

c) **i.** Find the minimum value of $|z_1 - z_2|$.

ii. Hence, find in exact cartesian form the values of z_1 and z_2 for which $|z_1 - z_2|$ has its minimum value.

d) **i.** Find in degrees, correct to one decimal place, the minimum value of $\text{Arg}(z_2)$

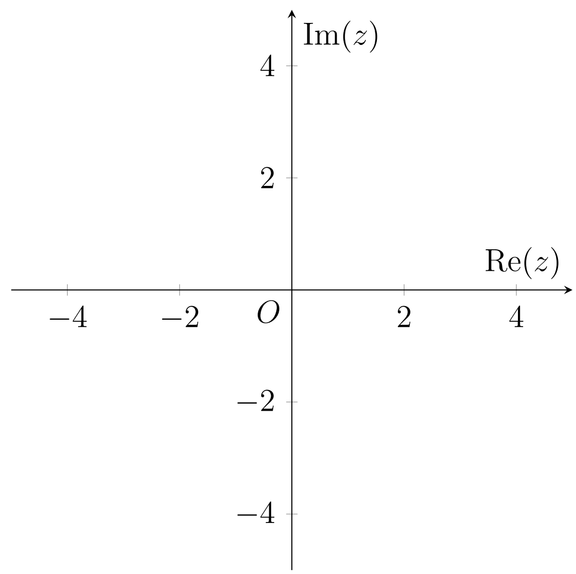
ii. Hence find the Cartesian form, correct to one decimal place, the value of z_2 for which $\text{Arg}(z_2)$ has its minimum value.

Question 9

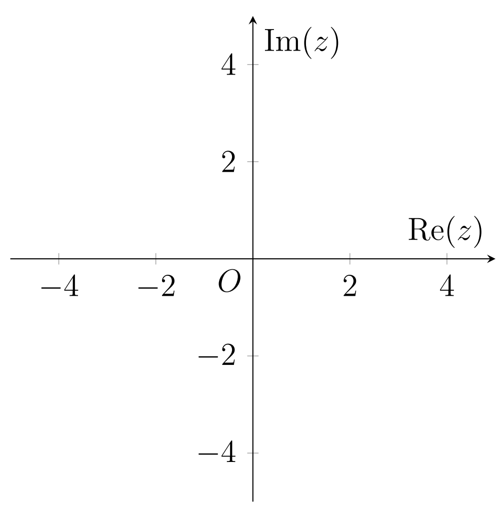
a) Consider the graph with the rule $\text{Arg}(z + 1 - i) = \frac{\pi}{4}$, where $z \in \mathbb{C}$

i. Write this region as a rule in Cartesian form.

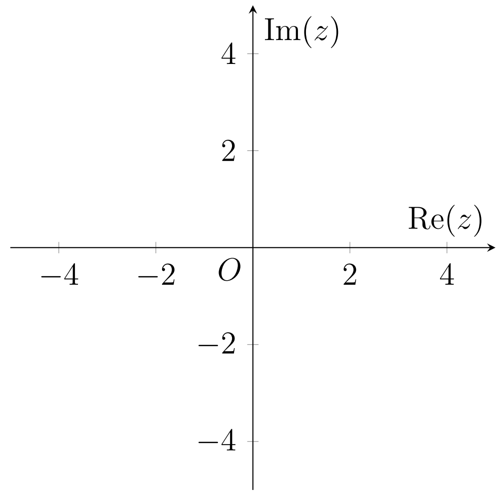
ii. Sketch this graph on the Argand diagram below.



iii. Shade the region $\{z : \text{Arg}(z + 1 - i) > \frac{\pi}{4}, z \in \mathbb{C}\}$ on the Argand diagram below.



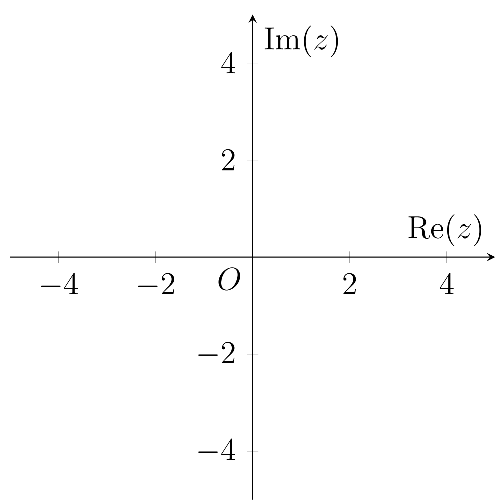
b) Sketch the region with rule $\text{Arg}((1+i)z+1-i) = \frac{\pi}{4}$, where $z \in \mathbb{C}$ on the Argand diagram below.



c) Consider the graph with rule $\text{Im}(2z+1-i) + \text{Re}(z+7+i) = 0)$, where $z \in \mathbb{C}$

i. Write this rule in Cartesian form.

ii. Sketch the graph with rule
$\{z : \text{Im}(2\overline{z}+1-i) + \text{Re}(z+7+i) = 0\} \cap \{\text{Arg}(z+1-i) > \frac{\pi}{4}\}$ on the Argand diagram below



Question 10

a) Let $w = 2 - 3i$ and $z_1 = 3 + 4i$

i. Find $\overline{w} + z$

ii. Find $|w|$

iii. Express $\dfrac{w}{z}$ in the form $a + bi$

b) On the Argand diagram, the complex numbers $O = 0, X = 1 + \sqrt{3}i, Y = \sqrt{3} + i$ and z_2 form a rhombus.

i. Find z_2 in the form $a + bi$, where a and b are real numbers.

ii. The interior angle, $\theta = \angle OXY$. Use a vector method to find the value of θ.

c) Find in polar form, all the solutions of $z^3 = 8$

d) **i.** Use the binomial theorem to expand $(\cos(\theta) + \sin(\theta)i)^5$

ii. Use de Moivre's theorem and your result from part (i) the prove that $\sin(5\theta) = 16\sin^5(\theta) - 20\sin^3(\theta) + 5\sin(\theta)$

iii. Hence, show that $x = \sin\left(\frac{\pi}{10}\right)$ is one of the solutions to $16x^5 - 20x^3 + 5x - 1 = 0$

iv. Find the polynomial $p(x)$ such that $(x-1)p(x) = 16x^5 - 20x^3 + 5x - 1$

v. Find the value of a such that $(4x^2 + ax - 1)^2$

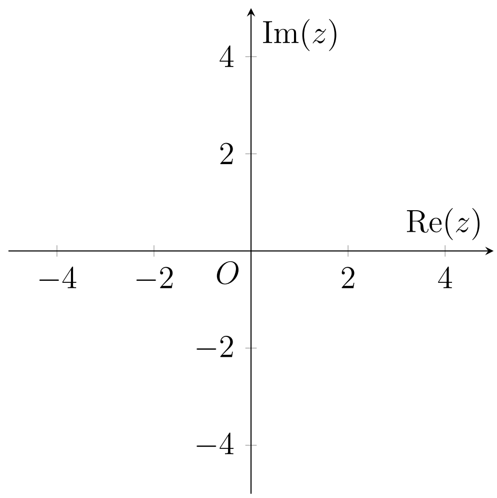
vi. Hence, find the exact value of $\sin\left(\frac{\pi}{10}\right)$

Question 11

Consider the graph with rule $|z + 1 - 2i| = 2$, where $z \in \mathbb{C}$

a) Write this rule in Cartesian form.

b) Sketch this graph on the Argand diagram below.



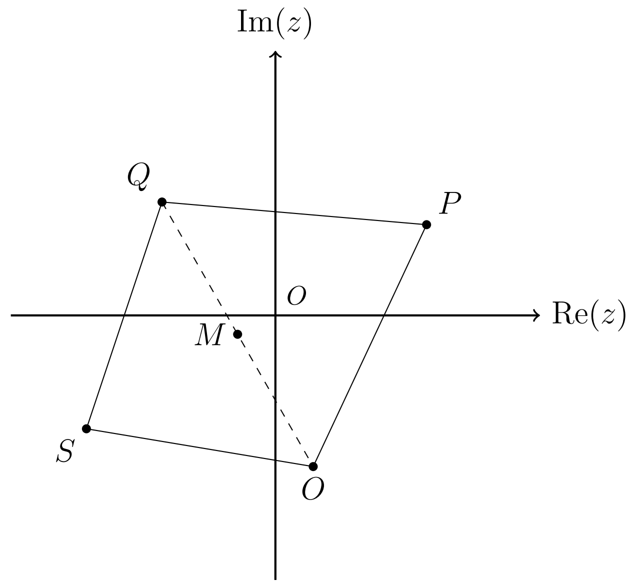
c) **i.** Consider the rule $\text{Arg}(z + i) = a$ where $0 < a < \frac{\pi}{2}$
Explain why the Cartesian equation for this rule is $y = mx - 1$,where $x > 0, m > 0$

ii. Hence, find in simplest form the exact minimum value of $\text{Arg}(z+i)$ where z satisfies $|z+1-2i|=2$

iii. Find in simplest form the exact value of z such that $\text{Arg}(z+i)$ has its minimum value.

Question 12

The point P on the Argand diagram below represents the complex number z. The points Q and R represent the points wz and $\overline{w}z$ respectively, where $w = \cos\left(\frac{2\pi}{3}\right) + \sin\left(\frac{2\pi}{3}\right)$. The point M is the midpoint of QR.
(The diagram is not drawn to scale.)



a) If $z = r\ \text{cis}(\theta)$ find wz and $\overline{w}z$ in polar form.

b) Hence explain why $|\overrightarrow{OP}| = |\overrightarrow{OQ}| = |\overrightarrow{OR}|$

c) Show that the complex number representing M is $-\frac{1}{2}z$

d) The point S is chosen such that $PQSR$ is a parallelogram. Find the complex number represented by S in terms of z.

Question 13

a) Use algebra to find the values which the real numbers a and b must take for $-2+i$ to be a solution to the complex equation $iz^2 - (2-ai)z + b + i - 0$.

b) Briefly explain why the conjugate root theorem cannot be used to find the second solution to this equation.

There are two different methods (parts **c.** and **d.** below) by which the second solution can be found.

c) **i.** Write down a linear factor of the polynomial $iz^2 - (2-ai)z + b + i$

ii. Use polynomial long division to find the second linear factor.

iii. Hence determine a second solution to $iz^2 - (2-ai)z + b + i = 0$.

d) **i.** Use the quadratic formula to show that two solutions of $iz^2 - (2 - ai)z + b + i = 0$ are:

$$z = \frac{2 - i + z_1}{2i} \text{ and } z = \frac{2 - i + z_2}{2i}$$

Where z_1 and z_2 are the two square roots of $7 + 24i$.

ii. Show that $7 + 24i$ can be expressed in the form $25\text{cis}(\theta + 2k\pi)$ where $\tan(\theta) = \frac{24}{7}$ and $k \in \mathbb{Z}$.

iii. Hence show that $z_1 = 5\text{cis}\left(\frac{\theta}{2}\right)$ and $z_2 = 5\text{cis}\left(\frac{\theta}{2} + \pi\right)$, where $\tan(\theta) = \frac{24}{7}$

e) **i.** Derive the identities $\cos^2\left(\frac{A}{2}\right) = \frac{1+\cos(A)}{2}$ and $\sin^2\left(\frac{A}{2}\right) = \frac{1-\cos(A)}{2}$

ii. Use the identities in **part a.** above to establish that $\cos\left(\frac{\theta}{2}\right) = \frac{4}{5}$ and $\sin\left(\frac{\theta}{2}\right) = \frac{3}{5}$

iii. Hence, obtain the values of z_1 and z_2 in Cartesian form.

iv. Hence, determine a second solution to $iz^2 - (2-ai)z + b + i = 0$.