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Specialist Maths Units 3/4

Differential Equations

Practice Questions

Short Answer Questions

Question 1

Find the values of k if $y = e^{kx}$ and $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 4y = 0$.

Question 2

Verify that $y = \log_e(\sin(2x))$ is a solution to the differential equation $y'' + (y')^2 + 4 = 0$.

Question 3

Euler's method, with a step size of 0.2, is used to approximate the solution of the differential equation $\sqrt{1+x^2}\,\frac{dy}{dx} = 1, y(0) = 1$. Find the approximate value for y when $x = 1$, correct to four decimal places.

Question 4

a) Water drops into an empty conical bucket at the rate of $60\text{cm}^3\text{s}^{-1}$. The radius of the base of the bucket is always equal to one quarter of its height, h at that point. Show that the rate at which the height of the water in the bucket is increasing is given by:

$$\frac{dh}{dt} = \frac{960}{\pi h^2}$$

b) Hence, express h in terms of t.

Question 5

Given that $\dfrac{dy}{dx} = \sqrt{4 - y^2}$ and $y = \sqrt{3}$ when $x = \dfrac{\pi}{6}$, find the particular solution for y.

Question 6

Suppose that a murder victim is found at 8.30 am and that the temperature of the body at that time is $30°C$. Assume that the room in which the murder victim lay was a constant $22°C$.
Suppose that an hour later the temperature of the body is $28°C$. Use this information to determine the approximate time the murder occurred.
We know that the normal temperature of a live human body is $37°C$.

Question 7

If $\frac{dP}{dt} = 0.05(1 - \frac{P}{1000})$ and $P(0) = 100$. Show that $P = \frac{1000}{9e^{-0.05t} + 1}$.

Question 8

If $y = xe^{-kx}$ is a solution of the differential equation $\frac{d^2y}{dx^2} = 4\frac{dy}{dx} - 4y$, find the exact value of k.

Question 9

Mr. Z has a pocket hand-warmer to keep his hands warm during a Richmond v Carlton game. Once activated, the hand warmer heats up at a constant rate of $16°C$ per minute. However, the hand warmer is also a subject to Newton's Law of Cooling. The outside temperature is $8°C$, the initial temperature of the hand-warmer is $8°C$ and stops heating when it reaches $72°C$.

a) Show that the rate of change of temperature (in $°C$) with respect to time (t minutes) is given by

$$\frac{dT}{dt} = \frac{72 - T}{4}$$

b) Find and simplify an expression for the time taken for the hand warmer to reach a temperature of $40°C$.

c) Find the temperature of the hand-warmer as a function of time.

Question 10

A freezer maintains a constant temperature of $-6°C$. A bottle of champagne that has a temperature of $24°C$ is placed in the freezer. Its temperature then drops by $4°C$ in one minute

a) Assuming Newton's Law of Cooling, show that

$$t = \frac{1}{\log_e(\frac{13}{15})} \log_e\left(\frac{T+6}{30}\right)$$

where T is the temperature of the bottle at t minutes after the bottle is placed in the freezer.

b) The optimal drinking temperature of champagne is $10°C$. Find the time taken to the nearest second for the champagne to reach the optimal drinking temperature.

c) The bottle will explode under pressure caused by cooling when its temperature is $4°C$. Find the time taken to the nearest second for the bottle of champagne to explode from when it has reached the optimal drinking temperature.

Question 11

The function $y = e^{-3x}(2x+1)$ is a solution of the differential equation

$$\frac{d^2y}{dx^2} + k\frac{dy}{dx} + my = 0, \text{ where } m \text{ and } k \text{ are constants.}$$

a) Show that $\dfrac{d^2y}{dx^2} = (18x-3)e^{-3x}$

b) Find the exact values of m and k.

Question 12

The number of bacteria in a colony grows according to the model $\dfrac{dN}{dt} = kN$, where N is the number of bacteria after t days. Initially there are 3000 bacteria. The number of bacteria was observed to increase to 7000 over the next 8 days.

a) Give the particular solution to the differential equation above, expressing N in terms of t.

b) How much longer will it take before there are at least 12000 bacteria in the colony ? Give your answer to the nearest day.

Question 13

Consider the differential equation $\frac{dy}{dx} = ye^{-x}$ with the initial condition $y(0) = 1$.
Use Euler's method with a step-size of 0.5 units to show that the solution to the differential equation at $x = 1$ is approximately $\frac{a\sqrt{e}+3}{b\sqrt{e}}$, where the ratio $\frac{a}{b}$ should be in exact form.

Question 14

A tank containing 400 litres of water has 10 kg of salt dissolved in it. A salt solution of concentration 0.2 kg/L is pumped in at a rate of 6L/min while the solution is allowed to run off at 6L/min.

a) Set up a differential equation for the amount of salt present, Q kg, at any time t.

b) By solving the differential equation, find the amount of salt in the tank after 20 mins.

c) If the solution is saturated when the concentration reaches 0.15kg/L, how long does it take to reach saturation point.

Question 15

A sailor, whose core body temperature is $37°C$, falls from a ship into icy waters of temperature $1°C$. The temperature of the sailor drops to $36°C$ in 5 minutes and he will not survive if his body temperature drops below $35°C$.

a) Assuming Newton's Law of Cooling, how long can the sailor survive in the water after he first falls in.

Two minutes after the sailor falls in,a ship is 200 metres away and a rescue boat is dispatched.

Due to seas and visibility, the boat travels to the sailor according to the differential equation $\dfrac{dx}{dt} = \dfrac{5}{\sqrt{t+1}}$, where x is the distance travelled by the boat towards the sailor in metres and t is the time in seconds.

b) Express x as a function of t.

c) How long has the sailor been in the water when the rescue boat arrives ?

d) Hence, decide whether the sailor is still alive or not.

Question 16

A vessel is in the form of an inverted circular cone with a semi-vertical angle of 30°. Water is poured in at $5\text{cm}^3/\text{min}$ and leaks out from the bottom at a rate of $\dfrac{\sqrt{h}}{30}$ cm^3/min, where h cm is the depth of the water at t min.

a) Set up the differential equation for the depth of the water at time t.

b) Sketch a suitable directional field for the differential equation and show the particular solution corresponding to the initial condition that $h(0) = 0.05$.

c) Set up a definite integral that gives the time for the depth to change from 0.5cm to 5cm. Find the time correct to 4 decimal places.

Question 17

If $y = -e^{-x}(\sin(x))$ is a solution to the differential equation

$$\frac{d^2y}{dx^2} + k\frac{dy}{dx} = 2e^{-x}\sin(x), k \in \mathbb{R}$$

Find the value of k.

Question 18

A metal bar is initially $900°C$ and cools to $800°C$ in 10 minutes when the surrounding temperature is kept at a constant $25°C$. Assume that Newton's Law of Cooling applies.

a) Solve an appropriate differential equation to find an expression for T, the temperature of the bar, in terms of t, the time in minutes.

b) At what time, in minutes correct to two decimal places, is the temperature of the bar $600°C$.

Question 19

Solve the differential equation $\dfrac{dy}{dx} = \dfrac{\sqrt{1+y^2}}{xy}$, given that when $x = 1, y = \sqrt{3}$.

Question 20

A tank initially contains 50L of brine in which 10kg of salt is dissolved. A brine solution containing 2kg/L of salt enters the tank at 5L/min, the mixture is then stirred and flows out at 3L/min.

a) Set up the differential equation for the amount of salt,, Q kg in the tank at time t.

b) Use Euler's method to find the amount of salt after 3 minutes, using a step size of 1 min.

c) If $Q(t) = 4(25+t) - K(25+t)^{-\frac{3}{2}}$ is a solution of the differential equation then K is found to equal $2(3^m 5^n)$.

Find m and n.

Question 21

a) A tank initially contains 500 litres of salt solution of concentration 0.02 kg/L. A solution of the same salt, but of concentration 0.05 kg/L, flows into the tank at the rate of 5 litres per minute. The mixture is kept uniform by stirring and the mixture flows out at the rate of 3 litres per minute. Let Q kg be the quantity of salt in the tank after t minutes. Set up the differential equation for Q in terms of t, specifying the initial condition.

b) If the solution to the differential equation found in part a. is given by

$$Q(t) = \frac{1}{20}(500 + 2t) + C(500 + 2t)^{-\frac{3}{2}}$$

Find the exact value of C.

Question 22

The solution to the differential equation $\frac{dy}{dx} = \frac{x\sqrt{x^2 - 1}}{e^{2y}}$, where $y = 0$ when $x = 1$, is given by $y = \log_e \sqrt{\frac{a}{b}(x^2 - c)^{\frac{b}{a}} + c}$ where a, b and c are integers. Find the values of a, b and c.

Question 23

The decrease in a thermometer's temperature reading is modelled by the differential equation

$$\frac{dT}{dt} = -k(T - T_0), \text{ where } k > 0, T_0^{\circ}C \text{ is the outdoor temperature.}$$

A thermometer is kept at a constant room temperature of $25°C$ is placed outside.
After 5 minutes outside the temperature is reading $15°C$.
Five minutes later the temperature is reading $10°C$.
If $T_0^{\circ}C$ is the thermometer's temperature reading t minutes after being placed outside, find T_0.

Question 24

At a particular school, it is thought that the rate at which a rumour spreads is jointly proportional to the number, x students, who have heard the rumour and the number, $N - x$ who have yet to hear it.

This relationship is modelled by the differential equation $\dfrac{dx}{dt} = kx(N - x)$.

a) Assuming that $x(0) = 1$, show that the particular solution to this differential equation can be written as

$$x = \frac{N}{1 + (N-1)e^{-Nkt}}$$

b) Eventually, (after a long time), what value does x approach ?

Multiple Choice Questions

Question 1

If $y = e^{kx}$ is a solution of the equation

$$\frac{1}{2}\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0$$

Where a, b are constants, then:

A. $k = 2a \pm \sqrt{a^2 - 2b}$

B. $k = -a \pm \sqrt{a^2 - 2b}$

C. $k = a \pm \sqrt{a^2 - 4b}$

D. $k = 2a \pm \sqrt{a^2 - 4b}$

E. $k = a \pm \sqrt{a^2 - 2b}$

Question 2

A cylindrical tank is filled with liquid to a depth of h metres so that the volume of liquid $V\text{m}^3$ in the tank is given by $V = 5\pi h$. Liquid flows into the tank at the rate of 0.1m^3 per hour and leaks out at the rate of $0.0\sqrt{h}\text{m}^3$ per hour. A differential equation that relates the variables h and t is:

A. $\dfrac{dh}{dt} = \dfrac{1}{50\pi} - \dfrac{1}{\sqrt{h}}$

B. $\dfrac{dh}{dt} = \dfrac{5 - \sqrt{h}}{250\pi}$

C. $\dfrac{dh}{dt} = \dfrac{5 - \sqrt{h}}{250\pi h}$

D. $\dfrac{dh}{dt} = 5\pi(0.1 - 0.02\sqrt{h})$

E. $\dfrac{dh}{dt} = \dfrac{0.1 - 0.02\sqrt{h}}{5\pi h}$

Question 3

The rate of change in a town's population depends on two factors. Immigration causes the population to rise by 6000 people per year, but poor water quality causes the death rate of 5% of the current population per year. A differential equation relating the population of the town N and t is:

A. $\dfrac{dN}{dt} = kN \quad k > 0$

B. $\dfrac{dN}{dt} = 0.05N + 6000$

C. $\dfrac{dN}{dt} = -0.05N + 6000$

D. $\dfrac{dN}{dt} = -0.05(N + 6000)$

E. $\dfrac{dN}{dt} = kN(6000 - N) \quad k < 0$

Question 4

If y satisfies the conditions $\dfrac{dy}{dx} = -50y$, and $y = 80$ when $x = 0$, then when $x = 1$.

A. $y = 80e^{-50}$

B. $y = e^{-50}$

C. $y = -80e^{-50}$

D. $y = -80e^{50}$

E. $y = 80e^{50}$

Question 5

A huge tank contains 100L of water when a salt solution of 3g/L is added at a rate of 10L/min. The mixture is kept uniform by stirring and flows out at 5L/min. If Q is the amount of salt in *grams* present in the tank at any time, t min, the Q satisfies the differential equation:

A. $30 - \dfrac{Q}{100}$

B. $30 + \dfrac{5Q}{100 - 5t}$

C. $30 - \dfrac{10Q}{100 + 5t}$

D. $30 - \dfrac{10Q}{100 - 5t}$

E. $30 - \dfrac{5Q}{100 + 5t}$

Question 6

Consider the differential equation $\dfrac{dy}{dx} = x^{\frac{3}{2}} + 2x$, with $x_0 = 1$ and $y_0 = 2$.
Using Euler's method with a step size of 0.1, the value of y_2, correct to three decimal places is:

A. 1.635

B. 2.635

C. 3.109

D. 2.300

E. 1.508

Question 7

A fish tank initially contains 4kg of salt dissolved in 100L of water. It is decided that this salt concentration is too high for saltwater fish to be kept, and so fresh water is mixed in at 10 per minute, while 10L of mixture is removed per minute.
If x kg is the concentration of the saltwater solution in the tank t seconds after the fresh water is first added. A differential equation for x would be.

A. $\frac{dx}{dt} = -\frac{x}{100}$

B. $\frac{dx}{dt} = -\frac{x}{10}$

C. $\frac{dx}{dt} = 10 - \frac{x}{100}$

D. $\frac{dx}{dt} = 10 - \frac{x}{10}$

E. $\frac{dx}{dt} = -\frac{x}{600}$

Question 8

Euler's method with a step size of 0.1 is used to find an approximate solution to the differential equation $\frac{dy}{dx} - \sin^{-1}(x - 2y)$ with the initial condition $y = 1$ when $x = 2$. When $x = 2.2$ the approximate value of y is:

A. 1

B. 1.1

C. 1.2

D. $1 + 0.1\sin^{-1}(0.1)$

E. $1.1 + 0.1\sin^{-1}(0.1)$

Question 9

The amount of a drug, x mg, remaining in a patient's bloodstream t hours after taking the drug is given by the different equation $\frac{dx}{dt} = -0.15x$.
The number of hours needed for the amount x to halve is:

A. $2\log_e\left(\frac{20}{3}\right)$

B. $\frac{20}{3}\log_e(2)$

C. $2\log_e(15)$

D. $15\log_e\left(\frac{3}{2}\right)$

E. $\frac{3}{2}\log_e(200)$

Question 10

The direction (slope) field for a particular first order differential equation is shown below. The equation that best represents this slope field is:

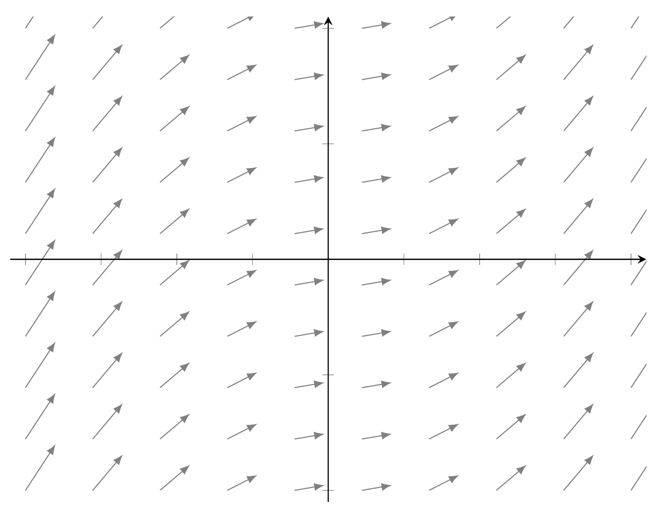
A. $\frac{dy}{dx} = \tan(x)$

B. $\frac{dy}{dx} = \sec^2(x)$

C. $\frac{dy}{dx} = |x|$

D. $\frac{dy}{dx} = \frac{1}{2}x^3$

E. $\frac{dy}{dx} = -2x^4$



Question 11

Let $\dfrac{dy}{dx} = 1 - \cos^{-1}(1 - x)$, and $(x_0, y_0) = (0, 1)$. Using Euler's method with a step size of 0.2, the value of y_2 is given by:

A. $1 - \cos^{-1}(0.8)$

B. $1 - 0.4\cos^{-1}(0.8)$

C. $1.4 - 0.2\cos^{-1}(0.8)$

D. $1.2 - 0.2\cos^{-1}(0.8)$

E. $1 - 0.2\cos^{-1}(0.8)$

Question 12

If $\dfrac{dy}{dx} = \sin\sqrt{x}$ and $y(0) = 2$, then $y(1)$ may be found by evaluating:

A. $\displaystyle\int_0^2 \sin\sqrt{x}\ dx + 1$

B. $\displaystyle 1 - \int_0^2 \sin\sqrt{x}\ dx$

C. $\displaystyle 2 - \int_0^1 \sin\sqrt{x}\ dx$

D. $\displaystyle\int_0^1 \sin\sqrt{x}\ dx - 2$

E. $\displaystyle\int_0^1 \sin\sqrt{x}\ dx + 2$

Question 13

The direction (slope) field for a differential equation is shown below. The equation of the differential equation could be:

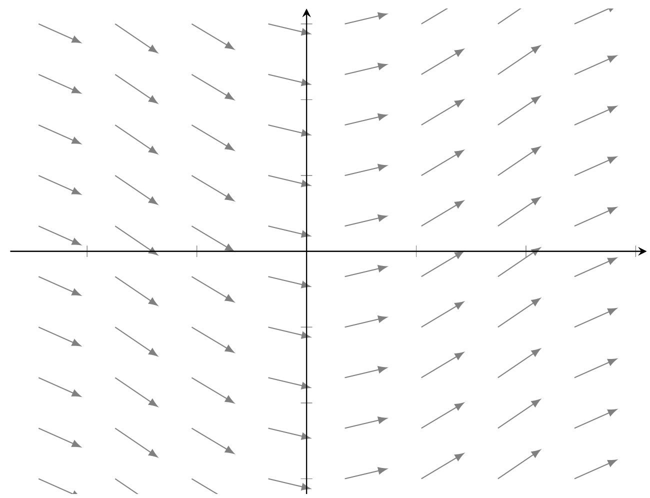
A. $\dfrac{dy}{dx} = -\sin(2x)$

B. $\dfrac{dy}{dx} = -\cos(2x)$

C. $\dfrac{dy}{dx} = \sin(2x)$

D. $\dfrac{dy}{dx} = \cos(2x)$

E. $\dfrac{dy}{dx} = \sin\left(\dfrac{x}{2}\right)$



Question 14

Water is leaking from a tank shaped like an inverted cone at a constant rate of $400\text{cm}^3/\text{min}$. The tank has a height of 30cm and a base radius of 5cm. Let h be the depth of the water in the cone at time t minutes. The rate of decrease of h in cm/min is:

A. $\dfrac{48}{\pi}$

B. $\dfrac{14400}{\pi h^2}$

C. $-\dfrac{100}{9\pi h^2}$

D. $-\dfrac{\pi h^2}{36}$

E. $\dfrac{100\pi h^2}{9}$

Question 15

A population of size x is decreasing according to the differential equation $\frac{dx}{dt} = -\frac{x}{250}$ where t denotes the time in days. The initial population size is x_0. The time in days taken for the population to halve is closest to:

A. 96

B. 132

C. 173

D. 207

E. 250

Question 16

A drum contains 100L of water in which 50g of salt is dissolved. Water containing 5g of salt per litre enters the drum at a rate of 6L per minute. The mixture in the drum is stirred continuously and flows out at a rate of 6L per minute.
A differential equation which models the amount $Q(t)$ of salt (in grams) in the tank at any time $t \geq 0$ is:

A. $\frac{dQ}{dt} = 30 - \frac{Q}{100}$

B. $\frac{dQ}{dt} = 30 - \frac{6Q}{100}$

C. $\frac{dQ}{dt} = -\frac{Q}{100}$

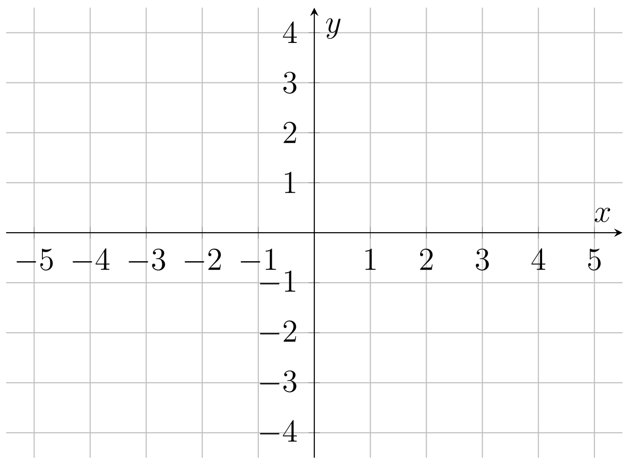
D. $\frac{dQ}{dt} = -\frac{6Q}{100}$

E. $\frac{dQ}{dt} = 30 - 6Q$

Extended Response Questions

Question 1

a) Sketch the direction field of the differential equation $\frac{dy}{dx} = 2(y-1)$ for $y \in \{-3, -2, -1, 0, 1, 2, 3\}$ at each of the values $x \in \{-3, -2, -1, 0, 1, 2, 3\}$ on the axes given below.



b) If $x = 1$ when $y = 0$, solve the differential equation given in **part a.** to find y in terms of x.

c) Sketch the graph of the solution curve found in **part b.** on the direction field in **part a.**.

d) The differential equation $\frac{dy}{dx} = \log_e\left(\frac{x+1}{x}\right)$ with the initial condition $x = 1, y = 1$ is to be solved using Euler's method with a step size of 0.5 units/ Express $y(2)$ in the form $a + \frac{1}{2}\log_e b$ where you should find the exact values of a and b.

Question 2

Suppose that a certain population of insects has a growth rate that varies jointly with time and the current population as:

$$\frac{dy}{dt} = \frac{1}{5}(0.5 + \sin(t))y$$

Where $y(t)$ is the population (in thousands) at time $t \geq 0$.

a) If $y(0) = 1$ find the first time, T, at which the population has doubled.

b) Choose two other initial conditions and determine whether the doubling time depends on the initial population.

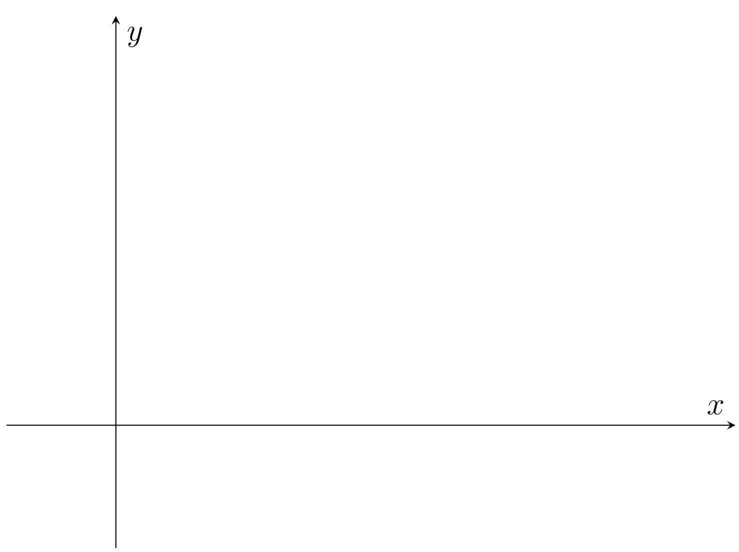
c) Suppose now that $\frac{1}{5}(0.5 + \sin(f))$ is replaced by its average value.

i. Show that the exact value of the average is 0.1.

ii. Determine the doubling time in this case.

d) Suppose that the $\sin(t)$ term in the differential equation is replaced by $\sin(2\pi t)$, that is, the variation in growth rate has a higher frequency. What effect does this have on the doubling time? Show all working.

e) Sketch the solutions obtained in **parts a, c** and **d.** on a single set of axes.



Question 3

The introduction of a certain mobile network in regional Victoria was initially taken up with enthusiasm by the population. After a certain period of time, most of the population has signed onto this network and the demand for new connections then tapered off. The rate if growth can be modelled by the differential equation $\frac{dy}{dt} = 80k(80 - y)$, where y describes the percentage taken up by the customer to the network and t, the time in months after the introduction of the network.

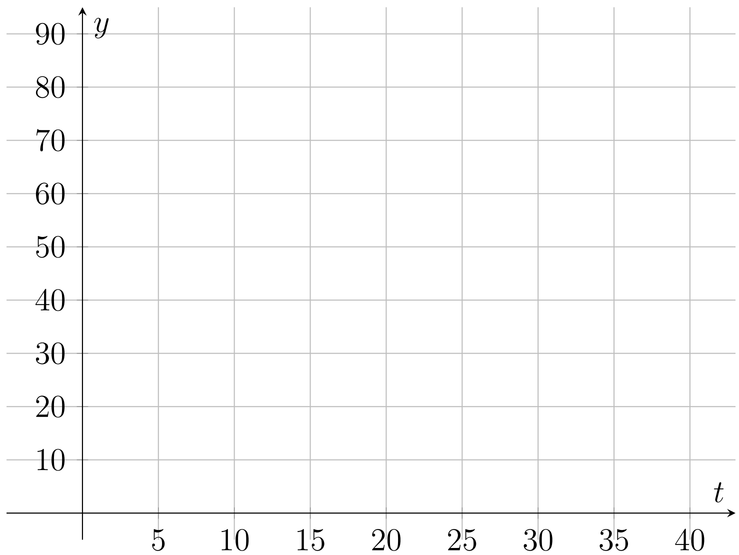
a) Verify that $y = 80 - e^{-80k(t-c)}$ is a solution to the differential equation $\frac{dy}{dt} = 80k(80 - y)$, where k and c are constants.

b) Given that initially only 1% of the population had taken up the connection of the network, find the expression for c in terms of the constant k.

c) After 10 months, 44% of the population had signed up. Using this information, show that the value of k can be written as:

$$k = -\frac{1}{800}\log_e\left(\frac{36}{79}\right)$$

d) Using the given value of k, determine the value c, correct to four decimal places. Hence, sketch the graph of the function $y = 80 - e^{-80k(t-c)}$ for $t \geq 0$ on the grid below.



e) From the information given in **part e.** state the percentage uptake of the network after 20 months. Given your answer to the nearest %.

Question 4

Newton's Law of Cooling states that the rate of cooling of a body is proportional to the difference between the temperature of the body, T, and that of its surroundings, θ, that is:

$$\frac{dT}{dt} = -k(T - \theta)$$

where k is a constant, and t is in hours. All temperatures are in degrees Celsius.

a) Show that $T = \theta + Ae^{-kt}$, where A is an arbitrary constant.

b) The police discover a dead body at 11.30am and estimates the body temperature of the corpse at $24°C$. At 12.00 midday, the police pathologist determines the temperature of the dead body to be $20°C$. The room temperature is a constant $17°C$.

i. Using **part a.** express the temperature of the dead body as a function of time and determine the exact values of θ, A and k.

ii. Given that the normal body temperature of a living person is $37°C$, what time did the person die. Give your answer to the nearest minute.

Question 5

A fish tank contains 50L of water, in which 100g of salt is dissolved. Jared needs to increase the salt level in the tank. He pours a salt solution containing 40g of salt per litre into the tank at a rate of 2L per minute, and simultaneously the well-stirred mixture leaves the tank at a rate of 2L per minute.

a) Show that an equation for Q, the amount of salt in the tank, at time t is

$$Q = 2000 - 1900e^{-\frac{t}{25}}$$

b) How much salt is in the tank after 5 minutes and what is the concentration then ? Answer correct to 3 decimal places.

c) Show analytically, the amount of salt which would be in the tank in the long term.

d) If the salt concentration in the tank exceeds 32g/L, the fish will die. After how long should Jared stop adding the solution, correct to the nearest minute.

Question 6

A tank initially contains 500L of brine which is a solution of water and salt. There is 100g of salt in the tank initially. Water is pumped into the tank at the rate of 60L per minute and brine is pumped out of the tank at the rate of 20L per minute. The amount of salt in the tank t minutes after the pumping begins is x grams. The solution in the tank is kept uniform by stirring.

a) **i.** Write down an expression for the concentration of salt in the tank, in grams per litre. In terms of x and t.

ii. Show that the differential equation which represents the rate of change of x with respect to t is given by $\frac{dx}{dt} + \frac{x}{25 + 2t} = 0$.

b) Show that $x(t) = \frac{500}{\sqrt{25 + 2t}}$.

c) Verify by substitution that $x(t)$ satisfies the differential equation $\frac{dx}{dt} + \frac{x}{25 + 2t} = 0$ and the initial conditions.

d) Find the amount of salt that flowed out of the tank in the first twenty minutes after the pumping begins. Give your answer in grams correct to one decimal place.

Question 7

Suppose that 50 rabbits are inadvertently introduced to a small island off the Victorian coast which was previously rabbit-free. The rabbits have no natural enemies on the island but due to food constraints the island can only support a population of 5000. The rate of growth of the rabbit population at time t months (after initial introduction) may be assumed to jointly vary with the number N present and the 'amount of room' $(5000 - N)$ available for additional rabbits. That is, the rate of growth is directly proportion to the product of N and $(5000 - N)$.
Note: Take one month to be exactly 30 days throughout the question.

a) If the initial rate of change of the population is 45 rabbits per month. Set up a differential equation involving N and t

b) **i.** Show that $t = \frac{11}{10}\log_e\left(\frac{N}{5000-N}\right) + c$, where c is constant.

ii. Calculate the value of c, showing your working, and hence express N as a function of t.

c) After what length of time, to the nearest day is the population growing most rapidly.

Question 8

A tank A initially contains 200 litres of pure water. A salt solution of concentration 400g/L is poured into tank A at a rate of 4L per minute.

a) **i.** If the solution flows out of a tank A at a rate of 3 litres per minute, show that

$$\frac{dM}{dt} = \frac{8t + 1600 - 15M}{5t + 1000}$$

where M kilograms is the amount of salt in tank A after t minutes. Assume a homogeneous solution.

ii. State the solution to the above differential equation.

In actual fact, the solution flows out of tank A and into tank B at a rate of 4 litres per minute.

b) **i.** Show that $\dfrac{dM}{dt} = \dfrac{80 - M}{50}$.

ii. Use calculus to solve the above differential solution, expressing M in terms of t.

iii. Explain what happens to the amount of salt and hence the concentration of the solution in tank A in the long run.

Tank B initially contains 200L of pure water. The solution of the salt and water in tank B flows out at the rate of 4L per minute.

c) **i.** Show that

$$\frac{dm}{dt} = \frac{80 - 80e^{-\frac{t}{50}} - m}{50}$$

where m kilograms is the amount of salt in tank B after t minutes. Assume the solution is homogeneous.

ii. The solution to the above differential equation is $m = 80 - 8e^{-\frac{t}{50}}(at + b)$. Use calculus to find the exact values of a and b.

d) Find, correct to the nearest minute, the time it takes for tank B to contain 20 kilograms of salt.

Question 9

Jessica is a well-known local identity who grows prized orchids. In order to improve the quality of her orchids, Jessica builds a greenhouse made of glass. The greenhouse traps heat from the sun, which helps to control the temperature at which the orchids grow at. In the absence of a heating system, the cooling of the greenhouse is modelled by the differential equation:

$$\frac{dT}{dt} = -k(T - T_a) \tag{1}$$

In this equation, $T°C$ is the temperature inside the greenhouse at time t hours after 6.00pm, k is a positive constant and $T_a°C$ is the temperature of the air outside the greenhouse. Assume T_a is constant from 6.00pm to 6.00am the following day.

After a sunny day the temperature inside the greenhouse is $20°C$ at 6.00pm and the outside temperature is $5°C$ from 6.00pm to 6.00am the following morning. Jessica observes that the temperature inside the greenhouse is $10°C$ at 9.00pm.

a) **i.** Use calculus to solve differential equation (1) and hence verify that $k = 0.3662$ (correct to 4 decimal places) for this greenhouse.

ii. Find (correct to two decimal places) the temperature inside the greenhouse at midnight.

iii. Find (correct to the nearest minute) the time when the temperature inside the greenhouse is $5.5°C$.

The orchids grow best when temperatures inside the greenhouse are kept at $10°C$ or above. In order to prevent the temperature from falling too low, Jessica has a heating system installed in the greenhouse. While the heating is switched on, it operates continuously and causes the temperature in the greenhouse to change according to the differential equation.

$$\frac{dT}{dt} = -k(T - T_a) + H \tag{2}$$

H is a positive constant relating to the available power of the heating system. The constants k and T_a are defined as earlier.
Use $H = 4, T_a = 5$ and $k = 0.3662$ for **parts b** and **c**.

b) **i.** Use calculus to solve differential equation (2).

ii. Find the temperature inside the greenhouse at 9.00pm, correct to two decimal places.

iii. Find the time at which the temperature inside the greenhouse is changing at the rate of $-1°C$ per hour, correct to the nearest minute.

c) **i.** Find the solution to the differential equation (2).

ii. Hence, find the values of T_0 (to 2 decimal places) for which the inside of the greenhouse would be expected to cool down, heat up or remain at a constant temperature.

The constant k relates to the heat losses from the greenhouse. Its value depends on a number of factors including the area of glass and the amount of insulation. Jessica decides to add more insulation to the greenhouse so that after a long time the temperature inside the greenhouse will approach $12°C$. If $H = 6$ and the outside temperature from 6.00pm to 6.00am the following morning is $-2°C$.

d) **i.** Find (correct to 4 decimal places) the value of k.

ii. If the temperature inside the greenhouse is $20°C$ at 6.00pm, write down a definite integral that gives the number of hours until the temperature inside the greenhouse reaches $12.5°C$. Hence, find the time when the temperature inside the greenhouse reaches $12.5°C$ correct to the nearest minute.