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Mathematical Methods Units 3/4

Functions and Relations

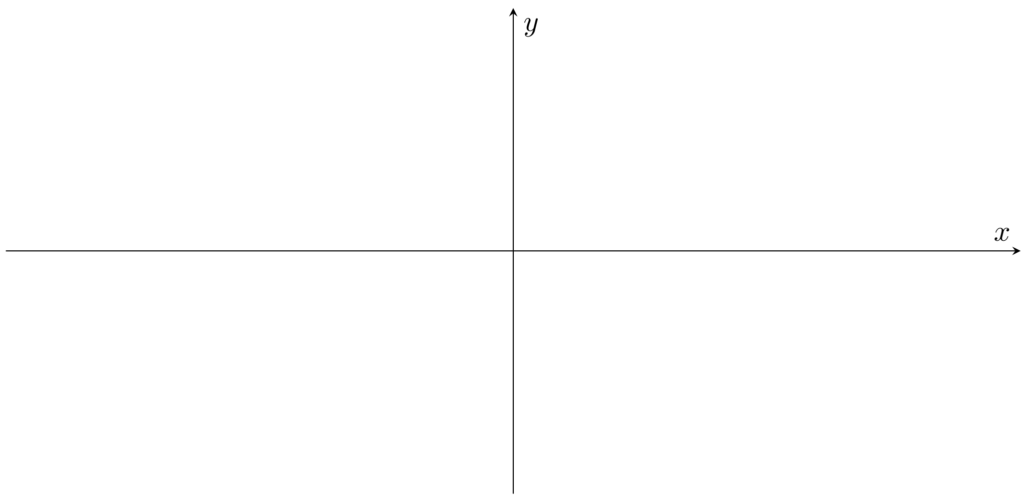
Practice Questions

- Functions and Relations
- Transformations
- Polynomial Functions
- Exponential and Logarithmic Functions
- Circular Functions

Short Answer Questions

Question 1

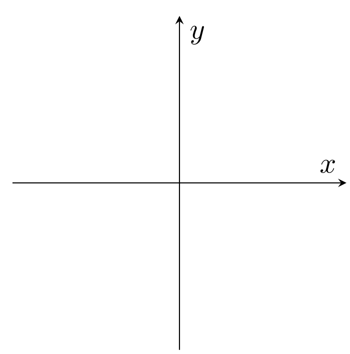
For the function $f(x) = -3e^{-x+1} + 2$, sketch the graph if $f(x)$ on the axis provided, clearly labelling all significant features.



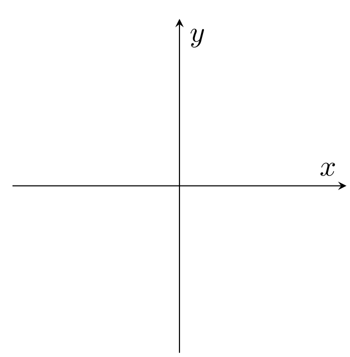
Question 2

Sketch the following functions for their maximal domain:

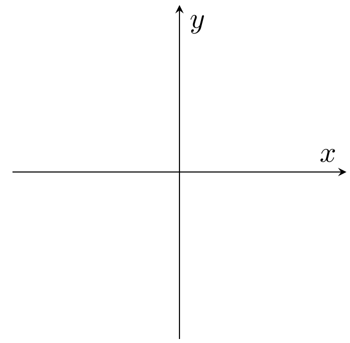
a) $f(x) = x^{\frac{17}{7}}$



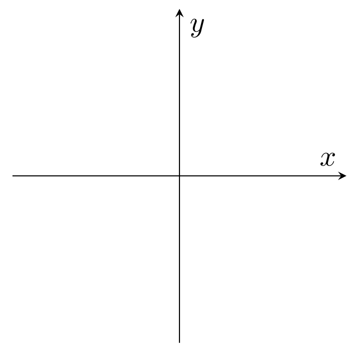
b) $f(x) = x^{\frac{5}{8}}$



c) $f(x) = x^{\frac{5}{17}}$



d) $f(x) = x^{\frac{-8}{5}}$



Question 3

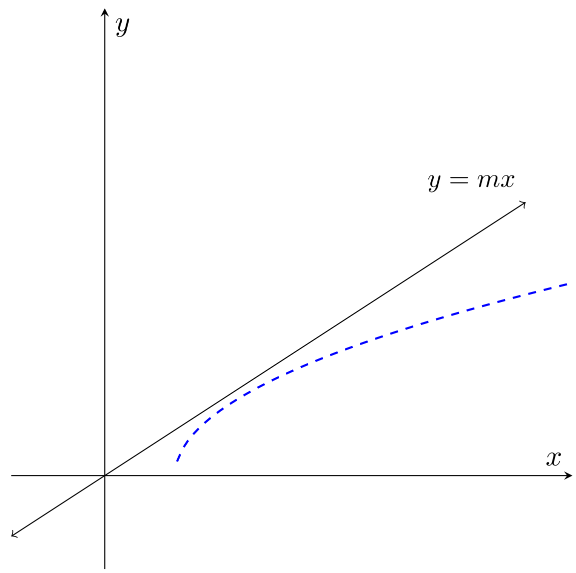
For the function $f : (-3, \infty) \to \mathbb{R}, f(x) = \dfrac{1}{(x+3)^2} + 1$

a) Find algebraically the inverse function $f^{-1}(x)$

b) Sketch both the function f and its inverse $f^{-1}(x)$ on the graph below. You must label all axis intercepts but do **NOT** label the point/s of intersection of these functions.

Question 4

a) Find the values of m for which the line $y = mx$ does **NOT** intersect the curve with equation $y = \sqrt{2x-3}$.



b) Let θ° be the angle formed by the line $y = mx$ and the positive direction of the x-axis. Find the values of θ° for which the line with equation $y = mx$ and the curve with equation $y = \sqrt{2x-3}$. intersect at least once.

Question 5

The points (3,10) and (5,12) lie on the graph of the function with rule $y = a\log_e(x-b)+c$. The graph has a vertical asymptote of $x = 1$.
Find the exact value of a, b and c.

Question 6

a) Find the general solution to the equation $\sqrt{3}\tan 3x = 1$.

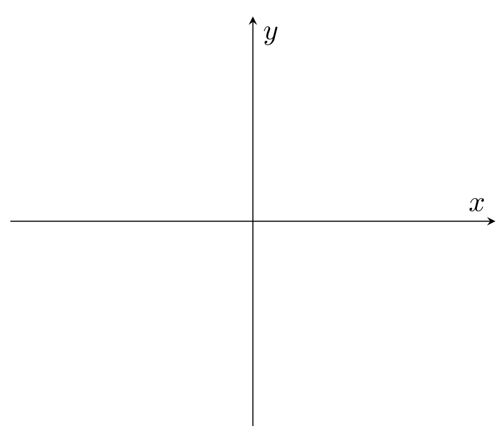
b) Hence or otherwise find all solutions in the interval $[0, \pi]$.

Question 7

Consider the function $f(x) = \dfrac{-x^3 + 8x^2 - 19x + 12}{x - 3}$.

a) Use long division to express $g(x)$ in the form $ax^2 + bx + c$ and hence state the values of a, b and c.

b) Sketch $g(x)$ on the set of axis provided below, labelling all important points.



Question 8

Find the exact x-intercept of the graph of $f(x) = x^3 - 3x^2 - 8x + 24$ for $x \geq 0$.

Question 9

a) Determine a transformation matrix in the form $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix}$ that transforms $f(x) = x^2$ to $g(x) = -2x^2 - 4x + 3$.

b) Find also, a transformation matrix that transforms $g(x)$ to $f(x)$.

Question 10

If the function f has the rule $f(x) = \sqrt{x^2 - 9}$ and the function g has the rule $g(x) = x + 5$.

a) Find the integers c and d such that $g(f(x)) = \sqrt{(x + c)(x + d)}$

b) State the maximal domain for which $f(g(x))$ is defined.

Question 11

If $\cos(x) = 0.75$, determine the value of $\sin\left(\frac{3\pi}{2} - x\right) + \cos(\pi + x)$.

Question 12

The graph of the function $f : [-2, 0] \to \mathbb{R}$ where $f(x) = e^x$ is reflected in the y-axis, translated 2 units to the left, then translated 3 units up. (Note the change in domain)

a) Specify the rule of the transformed graph.

b) Determine its exact range.

Question 13

Find the maximal domain of:

a) $P(x) = \sqrt{5 - x} + \sqrt{7 - 2x}$

b) $f(x) = \sqrt{16 - x^2}$

Question 14

Find the exact solution(s) of $e^x + 1 - 6e^{-x} = 0$.

Question 15

Find the value of x, in terms of a, for which $2\log_a x = 2 + \log_a 25$ where $a > 0$ and $x > 0$.

Question 16

a) Sketch the graph of $y = g(x)$ where $g(x) = -9 + \dfrac{1}{(x+2)^2}$

b) The function $g(x)$ is restricted to domain A so that:

- $g(x)$ is one to one.
- A includes $x = 3$.
- The domain of A is as large as possible.

i. Find this domain of A.

ii. Find the rule of $y = g^{-1}(x)$

iii. Find the domain of $g^{-1}(x)$.

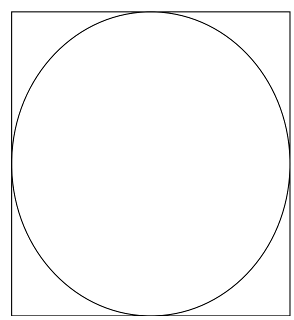
iv. Sketch the graph of $y = g^{-1}(g(x))$.

c) Write down the sequence of transformations that transform $y = \dfrac{1}{x^2}$ to $y = -9 + \frac{1}{(x+2)^2}$

Question 17

John is making a metallic cylinder. The base of the cylinder will be cut out from a square metal sheet as shown.

a) Find the area of the base of the cylinder in terms of x.



b) The height of the cylinder is 3 cm more than 4 times the length of the square. Find an expression for the volume of the cylinder.

Question 18

Let $f(x) = \sqrt{x+1}$ and $g(x) = \dfrac{1}{x^2}$.

a) State the implied domain of $f + g$.

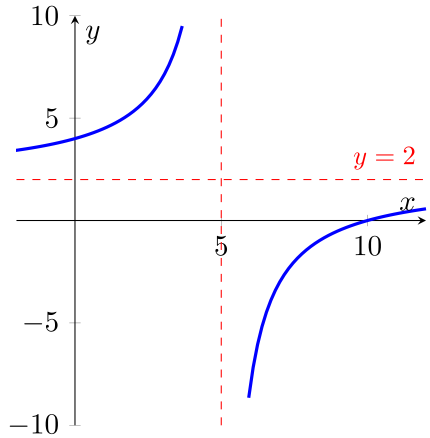
b) **i.** Find $f(g(x))$

ii. State the domain of $f(g(x))$

c) f_1 is a restriction of f such that $g(f_1(x))$ is defined. State the maximal domain of f_1.

Question 19

Determine the values of a, b and c in the equation $y = \dfrac{a}{x-b} + c$ in order to describe the graph below.



Question 20

The function t is defined by: $t : \mathbb{R}^- \cup \{0\} \to \mathbb{R}, t(x) = 4x^2 + 1$

a) State the domain and range of t.

b) Find the value of a such that $t^{-1}(x) = -\sqrt{\dfrac{x-1}{a}}$. State the range of t^{-1}.

Question 21

Solve $2^{3x+1} = 5^{x-a}$ for x (where $a \in \mathbb{R}$) giving your answer in the form $x = -\dfrac{\log_e A - a\log_e B}{\log_e C - \log_e D}$. Where A, B, C and D are integers.

Question 22

Express $y = \dfrac{4x-7}{x+2}$ in the form $y = p + \dfrac{t}{x+2}$ where p and t are integers.

Question 23

Given $f(x) = (x-1)^2$ and $g(x) = \sqrt{x} + 1$, find $g(f(x))$ in hybrid form.

Question 24

The height, h metres, above ground of a Ferris wheel seat is given by the rule $h(t) = 4 - 2\cos(\pi t)$, where t is the time in minutes after the wheel begins rotating.

a) How long does it take the wheel to complete one full revolution ?

b) How high off the ground is the seat after 20 seconds ?

An adjacent Ferris wheel starts rotating at the same time. The height, k metres above the ground of this Ferris wheel is given by $k(t) = 4 + 2\sqrt{3}\sin(\pi t)$, where t is the time in minutes after the wheel begins rotating.

c) Find the first two times that the two seats are at the same height.

d) Hence find the height at which the two seats are at the same height.

Question 25

A point with the coordinates (x, y) undergoes the following transformations:

- dilation factor 2 from the x axis.
- a reflection in the y axis.
- a translation 5 units in the negative direction of the x axis.

a) **i.** State that image (x', y') in terms of x and y respectively after the point has undergone these transformations.

ii. Hence or otherwise, state the image of the point (4, -2) after these sequence of transformations.

b) A function with the rule $y = x^3 + 2$ is mapped to the image $y = (4 + 5x)^3 - 3$. State an appropriate sequence of transformations for this mapping.

Question 26

Two functions p and q are defined such that:

$$p : \mathbb{R}^+ \to \mathbb{R}, p(x) = \frac{1}{x} + x \text{ and } q : (a, \infty) \to \mathbb{R}, q(x) = \frac{1}{x}$$

a) **i.** Find an expression for $\dfrac{p(x)}{q(x)}$ in simplest form.

ii. Determine the value of a for which the composite function $s(x) = p(q(x))$ exists.

iii. Write down the rule of $s(x)$ in terms of x.

iv. State the domain and range of $s(x)$.

b) Find the equation of the tangent to the graph $y = \frac{1}{x} + x$ at point p given that the equation of the normal line to the graph at point p is defined as $3y - x = 7$.

Question 27

When $q(x) = ax^2 + 3x - 4$ is divided by $(x - 2m)$ the remainder is -2. Find the value of a such that there is exactly one solution for m.

Question 28

Find the values of k for which the simultaneous equations $kx + (k+1)y = 1$ and $4x + 3ky = 2$ have a unique solution.

Question 29

State the sequence of transformations which takes:

a) the graph of $y = x^3$ to the graph of $y = 2(x-1)^3 - 3$.

b) the graph of $y = (3x-2)^3 + 3$ to the graph of $y = x^3$.

c) the graph of $y = x^3$ to the graph of $y = (2x-1)^3$.

Question 30

a) Solve the equation $-2\cos(x) = 1$ for $x \in [-2\pi, 2\pi]$

b) For the function $y = -2\cos(x)$, find when $\dfrac{dy}{dx} > 1$ for $x \in [-2\pi, 2\pi]$

Question 31

Let $f(x) = x^3 - 6$.

a) Is f an odd function, even function or neither.

b) Find $f^{-1}(x)$.

c) Find the point(s) of intersection between f and f^{-1}.

Multiple Choice Questions

Question 1

The relation $f(x) = \dfrac{1}{x-2} + 6, x \neq 2$ is best described as

A. a many to one function

B. a one to one function

C. a one to many function

D. a many to many function

E. an even function

Question 2

The diagram shows the graphs of $y = \log_a x$ and $y = \log_b x$. From the graphs, it can be concluded that.

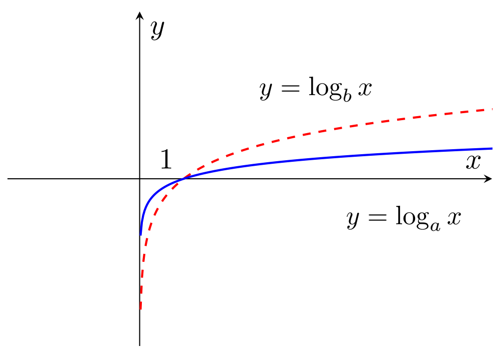
A. $a = b$

B. $a > b$

C. $a < b$

D. $a + b = 1$

E. $a = -b$



Question 3

Let $f(x) = \sqrt{6 + x - x^2}$ and $g(x) = \sqrt{-5x - 2}$. The maximal domain of $f(x) - g(x)$ is:

A. $3 - \sqrt{17} \leq x \leq 3 + \sqrt{17}$

B. $3 - \sqrt{13} \leq x \leq 3 + \sqrt{13}$

C. $-\frac{5}{2} \leq x \leq -2$

D. $-\frac{2}{5} \leq x \leq 3$

E. $-2 \leq x \leq -\frac{2}{5}$

Question 4

Given the function $f(x) = \frac{1}{(x+3)^2} - 4$, the inverse function would exist if the domain of $f(x)$ were:

A. $(-\infty, -2)$

B. $(-4, -3]$

C. $(-4, -2]$

D. $(-4, \infty)$

E. $(-3, -2)$

Question 5

Consider the polynomial $P(x) = x^3 - 6a^2x + ka^3$ where a and k are non-zero real constants. The polynomial has $x - 2a$ as a factor and when the polynomial is divided by $x + a$ the remainder is equal to -9 . Then:

A. $k = -4$ and $a = -1$

B. $k = -4$ and $a = 1$

C. $k = 4$ and $a = \sqrt[3]{9}$

D. $k = 4$ and $a = -1$

E. $k = 4$ and $a = 1$

Question 6

The transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by:

$$\mathrm{T}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

The equation of the image of the curve with equation $y = e^x$ under the transformation T is:

A. $y = 2e^{-x-2} - 2$

B. $y = 2e^{-(x-2)} + 2$

C. $y = 2e^{-x-2} + 2$

D. $y = \frac{1}{2}e^{-x-2} - 2$

E. $y = 2e^{-x+2} + 2$

Question 7

Consider the functions:

$$f : A \to \mathbb{R}, f(x) = \frac{-2}{\sqrt{2-3x}} \text{ and } g : B \to \mathbb{R}, g(x) = x^2 - 3x - \tfrac{5}{2}$$

The function $f(g(x))$ will be defined if the domain of g is:

A. $(-\infty, -1) \cup (4, \infty)$

B. $-1 \leq x \leq 4$

C. $\frac{2}{3} < x < \frac{3}{2}$

D. $\frac{9-\sqrt{195}}{6} < x < \frac{9+\sqrt{195}}{6}$

E. $\frac{9-\sqrt{195}}{6} < x < \frac{2}{3}$

Question 8

The graph of the function $f : [2, 5] \to \mathbb{R}, f(x) = \log_e(x)$ is reflected in the y-axis, translated 2 units to the left then translated 3 units up. The domain of the new graph is:

A. [-7, -1]

B. [-4, -1]

C. [-7, -4]

D. $[\log_e -7, \log_e -4]$

E. Not defined

Question 9

If $\sin(\theta) = \dfrac{5}{13}$ then $\cos(\theta)$ equals.

A. $\frac{\pi}{2} - \frac{5}{13}$

B. $\frac{\pi}{2} + \frac{5}{13}$

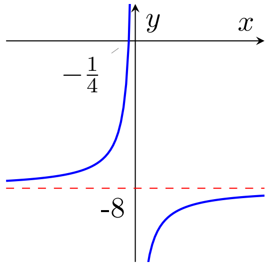
C. $\frac{12}{13}$

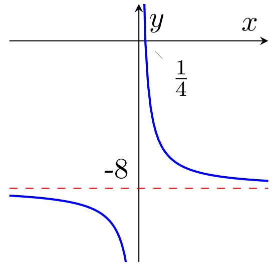
D. $\frac{5}{12}$

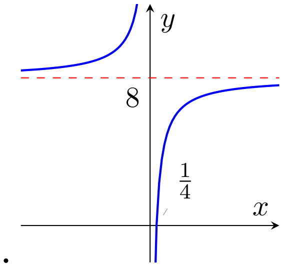
E. $\frac{5}{13}$

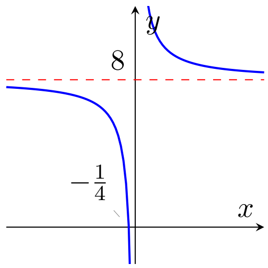
Question 10

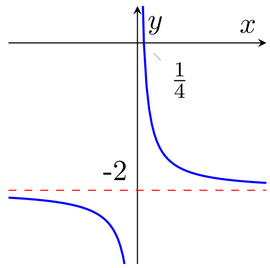
A sketch of the graph $8x + xy - 2 = 0$ is best shown by:

A. 

B. 

C. 

D. 

E. 

Question 11

The maximal domain for the function with the rule $f(x) = \dfrac{x}{\sqrt{6-2x}} - 1$ is:

A. $\mathbb{R} \setminus \{3\}$

B. $\mathbb{R}$

C. $(-\infty, 3]$

D. $(3, \infty)$

E. $(-\infty, 3)$

Question 12

The equation of the linear function that makes an angle of $\theta°$ with the positive direction of the x axis and passes through the origin would be:

A. $y = x$

B. $y = \theta \times x$

C. $y = \tan^{-1}\theta \times x$

D. $y = \tan\theta \times x$

E. $y = (180 - \tan\theta) \times x$

Question 13

The following transformations are applied to the graph of $y = e^x$ in the order:

- Dilation of factor 2 from the x-axis.
- Dilation of factor 2 from the y-axis.
- Horizontal translation of 4 units to the right.
- Vertical translation of 2 units down.
- Reflection in the x-axis.

The equation of the resultant graph is:

A. $y = -2e^{\frac{x}{2}-2} + 2$

B. $y = -2e^{\frac{x}{2}-2} - 2$

C. $y = -2e^{\frac{x}{2}-4} + 2$

D. $y = -2e^{2(x-4)} + 2$

E. $y = -2e^{2(x-4)} - 2$

Question 14

f us the function defined by $f(x) = \dfrac{1}{x^2+2}, x \in \mathbb{R}$.
A suitable domain restriction for the domain of f such that its inverse exists would be:

A. $[-1, 1]$

B. $\mathbb{R}$

C. $[-2, 2]$

D. $[0, \infty)$

E. $[-1, \infty)$

Question 15

The equations of the asymptotes of the graph with the rule $y = \dfrac{-3x - 1}{x + 1}$ are:

A. $x = 1$, $y = -1$

B. $x = 1$, $y = -3$

C. $x = -1$, $y = 0$

D. $x = -1$, $y = -3$

E. $x = -1$, $y = \frac{1}{3}$

Question 16

If $f(x) = f(-x), x \in \mathbb{R}$ then the graph of $y = f(x)$ is:

A. symmetric about the x-axis

B. symmetric about the y-axis

C. rotationally symmetric about the origin

D. discontinuous at the origin

E. Always increasing

Question 17

If a parabola has a turning point at $(z, 1)$, has a y-intercept at (0,-15) and a x-intercept is (5,0), then the value of z is:

A. 4

B. 0

C. 2

D. 8

E. 6

Question 18

Let $f : \mathbb{R} \to \mathbb{R}$, where $f(x) = e^x + k$, where k is a real constant. If f and f^{-1} has two points of intersection then:

A. $k < -1$

B. $k < 0$

C. $k > 1$

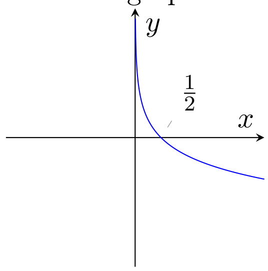
D. $k \leq 0$

E. $k \leq -1$

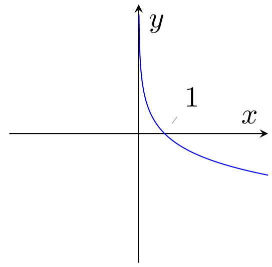
Question 19

The graph of $y = \log_e x$ is reflected in the x-axis and then dilated by a factor of $\frac{1}{2}$ in the x-direction. The resultant graph would look like:

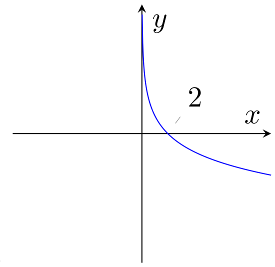
A.



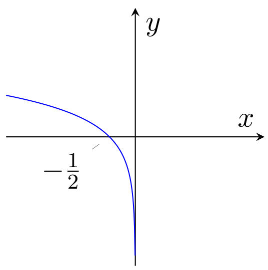
B.



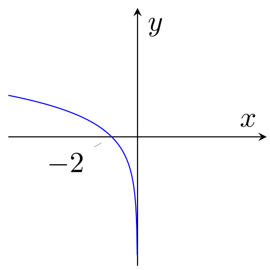
C.



D.



E.



Question 20

If function with rule

$$f(x) = \begin{cases} -x^2 + x, & x > -1 \\ -kx + 1, & x \leq -1 \end{cases}$$

is continuous then:

A. $k = -\frac{1}{3}$

B. $k = -3$

C. $k = 3$

D. $k = -1$

E. $k = 1$

Question 21

The inverse function $f^{-1}(x)$ is given by $f^{-1} : [5, \infty) \to \mathbb{R}, f^{-1}(x) = \frac{1}{3}(x-5)^2 + 3$.
The equation of $f(x)$ can be given by:

A. $f : [5, \infty) \to \mathbb{R}, f(x) = \sqrt{x-3} + 5$

B. $f : [3, \infty) \to \mathbb{R}, f(x) = \sqrt{3(x-3)} - 5$

C. $f : [3, \infty) \to \mathbb{R}, f(x) = \sqrt{3(x-3)} + 5$

D. $f : [3, 5) \to \mathbb{R}, f(x) = \sqrt{5(x-3)} + 3$

E. $f : [5, \infty) \to \mathbb{R}, f(x) = \sqrt{3(x-3)} + 5$

Question 22

If the curve with rule $y = f(x)$ is transformed by

- A translation of a units in the positive direction of the x-axis followed by
- A translation of b units in the positive direction of the y-axis followed by
- A Dilation by a factor of 3 from the y-axis,

The transformed curve will have the rule:

A. $3f(x+a) + 3b$

B. $3f(x-a) + 3b$

C. $f(\frac{x}{3} + \frac{a}{3}) + b$

D. $f(\frac{x}{3} + a) + b$

E. $f(3x - a) + b$

Question 23

Given the function $f : [0, 4b) \to \mathbb{R}, fx) = -x^2 + 2bx + 3b^2$ where $b \in \mathbb{R}^+$,then the range of the function of f is:

A. $[3b^2, 4b^2)$

B. $(3b^2, 4b^2]$

C. $(3b^2, 5b^2]$

D. $(-5b^2, 3b^2]$

E. $(-5b^2, 4b^2]$

Question 24

The graph of the function with equation $y = ae^{-x} + b$ is shown below. The values of a and b respectively are:

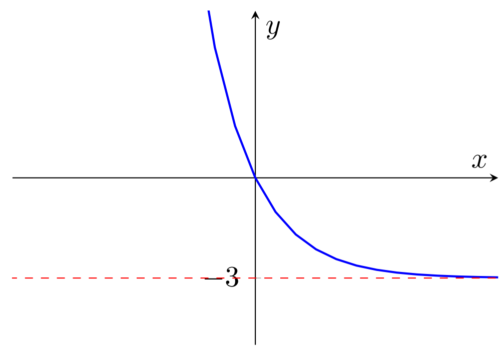
A. $x = 3$ and $y = -2$

B. $x = 2$ and $y = -3$

C. $x = -2$ and $y = 3$

D. $x = 3$ and $y = -2$

E. $x = -3$ and $y = -2$



Question 25

Consider the functions:

$$f : (-2, \infty) \to \mathbb{R}, f(x) = \frac{1}{(x+2)^2} \text{ and } g : (a, \infty) \to \mathbb{R}, g(x) = x - 1$$

The smallest value of a such that the composite function $f(g(x))$ exists is:

A. 2

B. 1

C. 0

D. -1

E. $-\infty$

Question 26

Which one of the following is a complete set of linear factors of the third degree polynomial $ax^3 - bx$, where a and b are positive real numbers ?

A. $x, ax^2 - b$

B. $x, ax - b, ax + b$

C. $x, \sqrt{ax - b}$

D. $x, \sqrt{a}x - b, \sqrt{a}x + b$

E. $x, \sqrt{a}x - \sqrt{b}, \sqrt{a}x + \sqrt{b}$

Question 27

The graph shown could be that of a function f whose rule is:

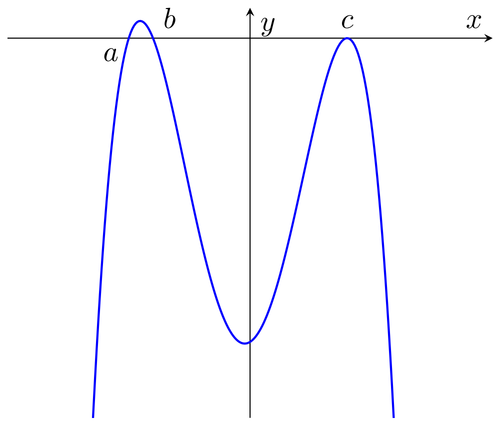
A. $(x-a)(x-b)(x-c)^2$

B. $(x+a)(x+b)(x-c)^2$

C. $(x-a)(x-b)(x-c)$

D. $(x-a)(b-x)(x-c)^2$

E. $(x-a)(x-b)(c-x)^2$



Question 28

The graph of the truncus $y = g(x)$ is shown below.
The graph of $y = g^{-1}(x)$ has asymptotes at

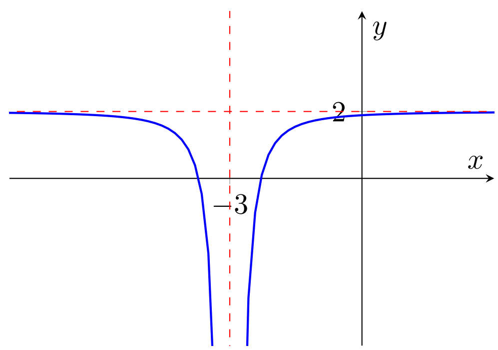
A. $x = 3$ and $y = -2$

B. $x = 2$ and $y = -3$

C. $x = -2$ and $y = 3$

D. $x = 3$ and $y = -2$

E. $x = -3$ and $y = -2$



Question 29

It is known that the graph with rule $y = 2ax + \cos(2x)$ has an x-axis intercept when $x = \pi$. The value of a is:

A. 2

B. $\frac{1}{2\pi}$

C. 2π

D. -2π

E. $-\frac{1}{2\pi}$

Question 30

If $f(x) = g(x) + h(x)$ and $g(x)$ is defined for $x > -3$ and $h(x)$ is defined for [-3, 2] then the domain of $f(x)$ is:

A. (-3, 2]

B. [-3. 2)

C. [-3, 2]

D. (-3, 2)

E. (-3, ∞)

Question 31

The function $f : [a, b] \to \mathbb{R}$ where $f(x) = x^2 - x^4$ has an inverse function f^{-1} if

A. $a = -1$ and $b = 0$

B. $a = -1$ and $b = 1$

C. $a = -\frac{1}{\sqrt{2}}$ and $b = 0$

D. $a = -\frac{1}{\sqrt{2}}$ and $b = \frac{1}{\sqrt{2}}$

E. $a = 0$ and $b = 1$

Question 32

The changes required to transform the graph of $y = \log_e(x)$ into the graph of $y = \frac{1}{2}\log_e(4 - 2x)$ are:

A. Dilation of factor 2 from the x-axis, Dilation of factor $\frac{1}{2}$ from the y-axis, Reflection in y-axis and Translation of 2 units to the right.

B. Dilation of factor $\frac{1}{2}$ from the x-axis, Dilation of factor $\frac{1}{2}$ from the y-axis, Reflection in x-axis and Translation of 2 units to the right.

C. Dilation of factor $\frac{1}{2}$ from the x-axis, Dilation of factor $\frac{1}{2}$ from the y-axis, Reflection in y-axis and Translation of 2 units to the right.

D. Dilation of factor 2 from the x-axis, Dilation of factor $\frac{1}{2}$ from the y-axis, Reflection in y-axis and Translation of 2 units to the left.

E. Dilation of factor 2 from the x-axis, Dilation of factor $\frac{1}{2}$ from the y-axis, Reflection in x-axis and Translation of 2 units to the right.

Question 33

The maximal domain of the function $f(x) = \sqrt{\dfrac{3-x}{2}}$ is:

A. $\mathbb{R}$

B. $\mathbb{R}^+$

C. $[3, \infty)$

D. $(-\infty, 3]$

E. $(-\infty, \frac{3}{2}]$

Question 34

The equations of the asymptotes of the graph with the rule $y = \dfrac{5x-6}{x+3}$ are:

A. $x = -3$, $y = 5$

B. $x = 3$, $y = 4$

C. $x = 3$, $y = 4$

D. $x = 3$, $y = 5$

E. $x = -3$, $y = \frac{6}{5}$

Question 35

If $[0, \infty) \to \mathbb{R}, f(x) = \sqrt{x}$ and $g(x) = -(x-3)(x+4)$ then the largest possible domain of g for which the composite function $h(x) = f(g(x))$ is defined is:

A. $(-\infty, -4] \cup [3, \infty)$

B. $[-4, 3]$

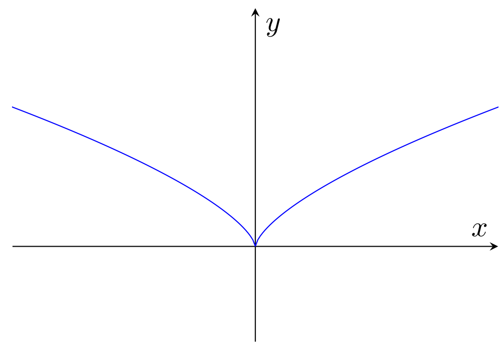
C. $(-\infty, -4) \cup (3, \infty)$

D. $(-4, 3)$

E. There is no possible domain for g such that $h(x)$ is defined.

Question 36

The rule of the curve sketched below is:



A. $x^{\frac{1}{2}}$

B. $x^{\frac{2}{3}}$

C. $x^{\frac{3}{5}}$

D. $-x^{\frac{1}{2}}$

E. $x^{\frac{4}{3}}$

Question 37

The asymptote(s) on the graph of $y = A - \dfrac{B}{C - x}$ are

A. $x = -C$

B. $x = -C$ and $y = -A$

C. $x = C$ and $y = A$

D. $x = -C$ and $y = A$

E. $x = C$ and $y = B$

Question 38

How many solutions does the equation $\cos^2(x) = \frac{1}{2}$ have for $0 \le x \le \pi$?

A. 2

B. 3

C. 1

D. 4

E. 0

Question 39

The velocity of an oscillating mass attached to a spring is given by $v(t) = 4 - 5\cos\left(\frac{\pi}{30}t\right)$, where t is the time in seconds after the mass begins the move. The mass is first stationary after approximately:

A. 1.6 seconds

B. 4.8 seconds

C. 6.1 seconds

D. 6.7 seconds

E. 2.6 seconds

Question 40

The solutions to $e^{2x} + ae^{x} + b = 0$ are $x = 0$ and $x = \log_e 2$. The value of a is:

A. 3

B. 2

C. $-3, 2$

D. -2

E. -3

Question 41

A possible equation of the graph shown on the right is:

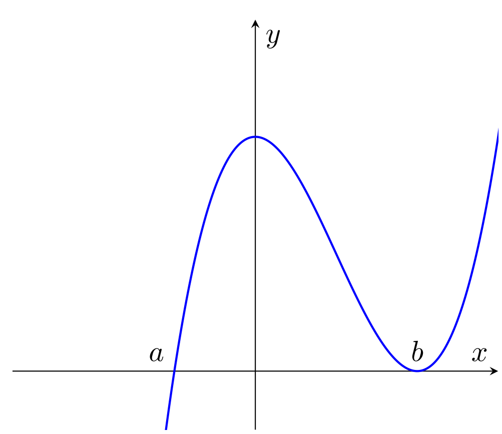
A. $f(x) = (x - a)(x - b)^2$

B. $f(x) = (x + a)(x - b)^2$

C. $f(x) = (x - a)(x + b)^2$

D. $f(x) = (x + a)(x + b)^2$

E. $f(x) = (x - a)^2(x - b)$



Question 42

The function $f(x) = \dfrac{x}{x^2+3}$ has a turning point at $\left(-\sqrt{3}, \frac{-\sqrt{3}}{6}\right)$
It follows that the function $g(x) = 4 - f(3-2x)$ has a turning point at:

A. $\left(\frac{3+\sqrt{3}}{2}, \frac{24+\sqrt{3}}{6}\right)$

B. $\left(\frac{6+\sqrt{3}}{2}, \frac{24+\sqrt{3}}{6}\right)$

C. $\left(\frac{\sqrt{3}-6}{2}, \frac{24+\sqrt{3}}{6}\right)$

D. $\left(2\sqrt{3}-3, \frac{\sqrt{3}-24}{6}\right)$

E. $\left(2\sqrt{3}+3, \frac{24+\sqrt{3}}{6}\right)$

Question 43

The function $f : [a, \infty) \to \mathbb{R}$, with rule $f(x) = \log_e(x^4)$, will have an inverse function if:

A. $a < 0$

B. $a \leq 0.1$

C. $a \leq 1$

D. $a > 0$

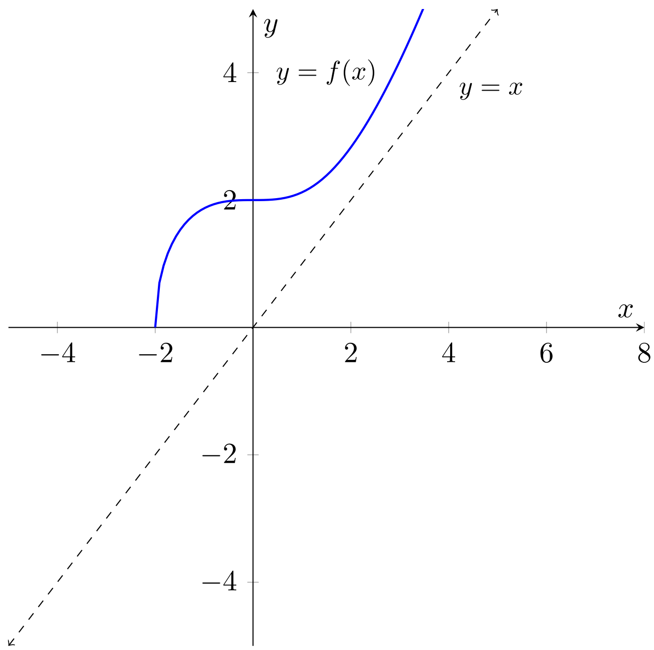
E. $a \geq 0.1$

Extended Response Questions

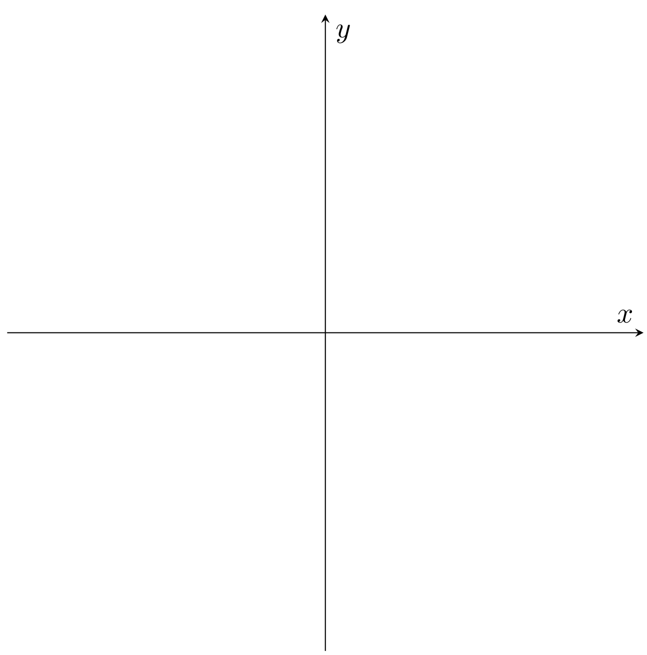
Question 1

Consider the function $f : [-2, \infty) \to \mathbb{R} = \sqrt{\dfrac{x^3+8}{2}}$

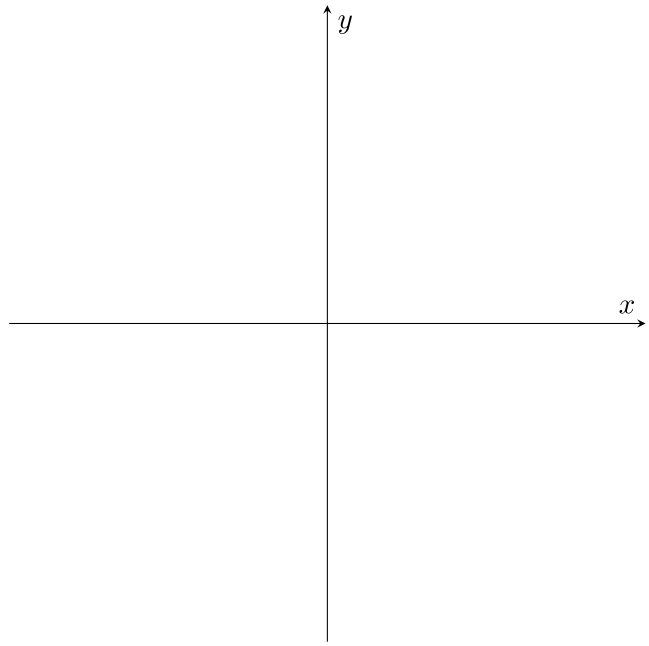
a) Sketch the graph of $y = f^{-1}(x)$ on the axes below, labelling all the coordinates of all axis intercepts.



b) Sketch the graph of $y = 3f(-x)$ on the axes below, labelling all the coordinates of all axis intercepts.



c) Sketch the graph of $y = f^{-1} \circ f(x)$ on the axes below.



d) Find the rule of f^{-1}, the inverse of f.

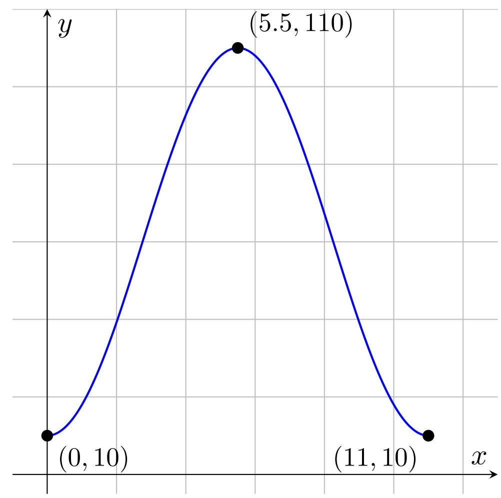
e) Find the values of a and b given a dilation by a factor of $\frac{1}{4}$ from the y axis maps the graph of $y = \sqrt{\dfrac{x^3+8}{2}}$ to the graph of $y = a\sqrt{bx^3+1}$

Question 2

The number of sunspots during a solar cycle is modelled with the function:

$N : [0, 11] \to \mathbb{R}, N(t) = b - a\cos(nt), \quad a, b, c \in \mathbb{R}^+$

Where N is the number of sunspots t years after the start of the cycle.



a) According to this model:

i. How many solar cycles occurred between 1755 and 2008?

ii. What is the range of N?

iii. Show that $n = \dfrac{2\pi}{11}$

iv. Show that $a = 50$ and $b = 60$

Assume that a new cycle began on January 1 2009

b) What is the predicted number of sunspots on January 1 2011, correct to the nearest integer?

c) The level of UV radiation increases with the number of sunspots. Bryan proposes to monitor UV radiation levels when $N \geq 80$.
For what length of time in each period is $N \geq 80$? Express your answer to the nearest month.

Bryan's colleague, Mariam, noted that the historical data shows considerable variation in the amplitude of the solar cycle. For the current cycle, she proposes that alternative model,

$$M : [0, 11] \to \mathbb{R}, M(t) = 60 - 50e^{-kt}\cos\left(\frac{2\pi t}{11}\right)$$

Where k is a real constant.

d) If Mariam's model predicts that there will be 89 sunspots in January 1 2014, show that $k = 0.10$ correct to two decimal places.

e) Find the exact points of intersection of the graphs M and N.

Question 3

a) The number of deer in a park after t years can be predicted by the model:

$$N(t) = 120(1.05)^t$$

i. State the initial number of deer in this park.

ii. State the number of deer in this park after 5 years.

iii. After how many years will the number of deer in this park be double the initial number? Give your answer to the nearest year.

The number of deer in a different park can be modelled by an equation of the form:

$$M(t) = ab^t$$

On this year the number of deer increases at a rate of 14% per year and there are 173 deer after 5 years.

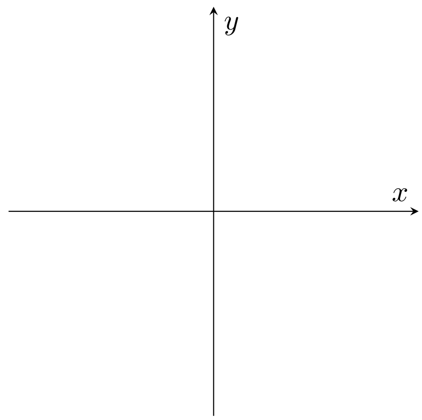
b) State the value of b and show that the value of a is 90. Hence state the equation for the number of deer in the second park.

c) At what time, correct to one decimal place, do both farms have the same number of deer?

Question 4

a) Consider the curve $y = -\dfrac{1}{x-2} - 1$

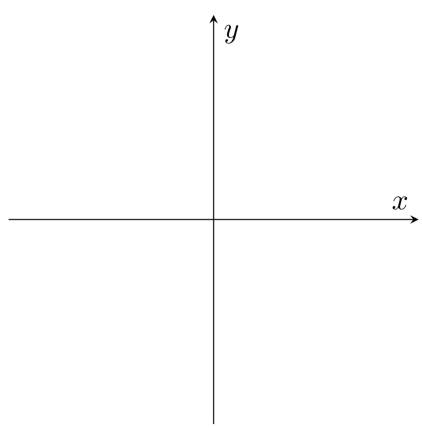
i. Sketch the curve on the axes provided, include all key features.



ii. Find the rule for the inverse function.

iii. Find the coordinates of the intersection points of $y = f(x)$ and $y = f^{-1}(x)$.

iv. Sketch $y = f \circ f^{-1}$ on the axes provided, include all key features.



b) Consider the curve $g(x) = -\dfrac{3}{x-2} + 2m$.
Find the value(s) of m so that the curve $g(x)$ intersects exactly once with the curve $y = g^{-1}(x)$

Question 5

The air pressure P (in cm of a mercury column) at a height h km above sea level where $h \geq 0$ is modelled by the equation:

$$P = p_0 e^{-kh}$$

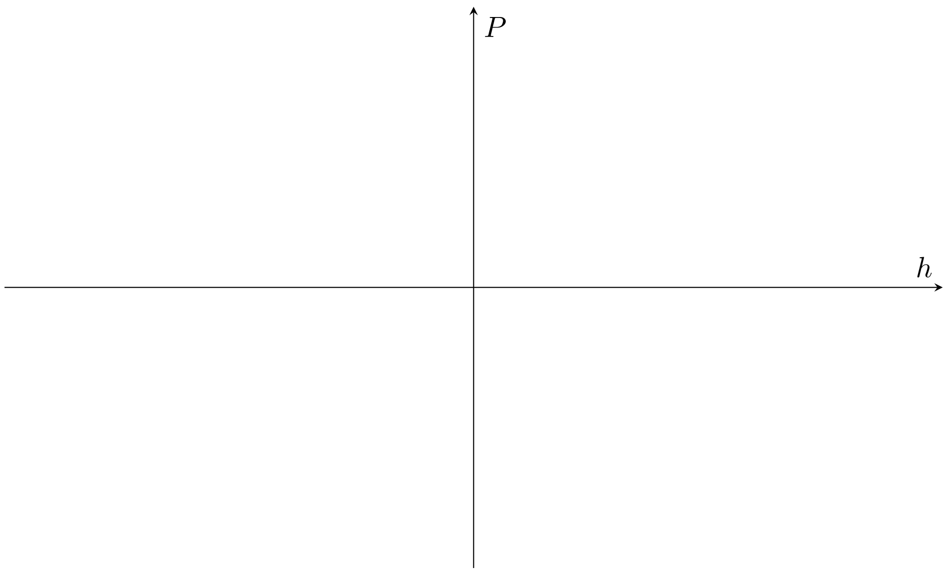
Where p_0 is the pressure at sea level and k is a constant. The air pressure at heights of 3km and 6km are 30cmHg and 20cmHg respectively.

a) Use algebra to show that the value of k is $\frac{1}{3}\log_e(1.5)$.

b) Hence, show that the value of p_0 is 45.

c) Calculate the air pressure, correct to 1 decimal place, in cmHg at a height of 5km.

d) Sketch the graph of P against h labelling all key features.



e) Find the rule of the inverse function for $y = 45e^{-\frac{1}{3}\log_e(1.5)x}$ writing the answer in the form

$$y = \frac{\log_e x - \log_e a}{b \log_e c}$$

Question 6

Solar park decides that it wants to have a pet enclosure.
From a previous job that was done they have 500m of fencing available.

a) A scenario is to make a rectangle.

i. Given that the width of the enclosure is to be $8x$ metres, state the length of the rectangle in terms of x.

ii. Show that area of the rectangle is given by $A(x) = 2000x - 64x^2$.

iii. State an appropriate domain for $A(x)$.

iv. What is the maximum area of the rectangle ?

b) Given that the sane amount of fencing is available, what is the maximum area that could be enclosed by a circular enclosure. (Give your answer to the nearest m^2)

c) A third option is an isosceles triangle with 2 sides that are each $2x$ metres long. Given that there is 500m of fence available.

i. State the length of the third side length in terms of x metres.

ii. State the possible values that x can take (in metres).

iii. The height of the triangle is given by $h = [500(2x - p)]^{\frac{1}{2}}$. Find the value of p.

iv. Find the maximum area for this triangle. (Give your answer to the nearest m^2.)

d) The boss says they need at least 16000 m^2 to fit all the animals in. Which of the three models could be used? Explain with justification.

e) State the sequence of transformations that map $h = x^{\frac{1}{2}}$ to $h = [500(2x - p)]^{\frac{1}{2}}$.

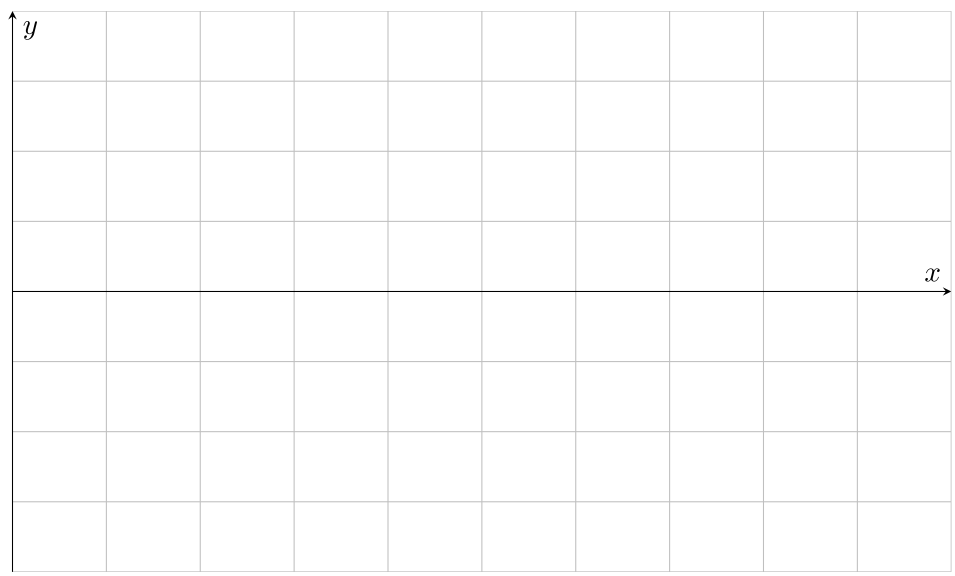
Question 7

According to Fitt's Law, for a fixed distance travelled by the mouse, the time take, in seconds, is given by $a - b\log_e(x), 0 < x \leq 5$ where x cm is the button width and a and b are positive constants for a particular user.

a) Minnie discovers that for her, $a = 1.1$ and $b = 0.5$.

i. Let $: (0, 5] \to \mathbb{R}, f(x) = 1.1 - \frac{1}{2}\log_e(x)$.

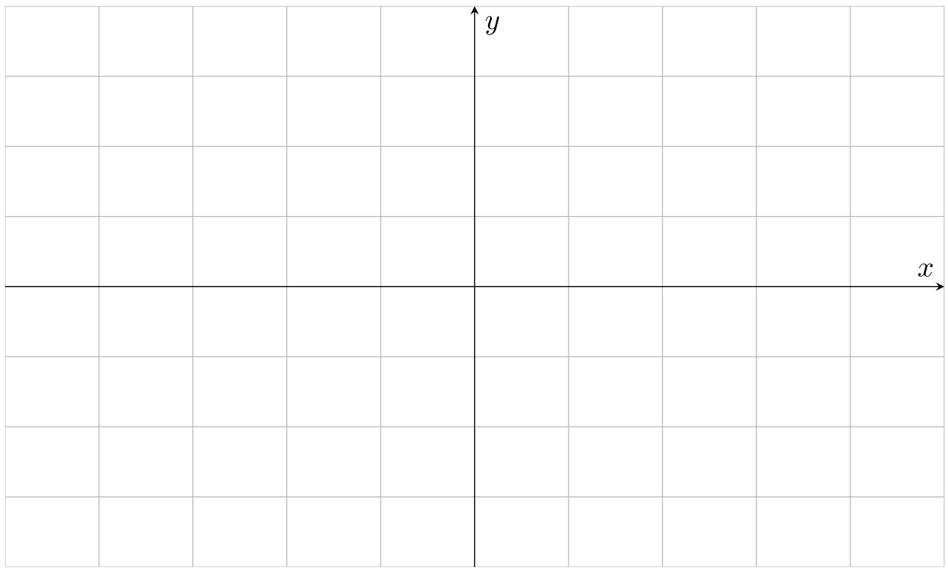
Sketch the graph of $y = f(x)$ on the axes below. Label any asymptote with its equation and any intercepts and end-point/s with their exact coordinates.



ii. Explain why f^{-1}, the inverse function of f, exists.

iii. Find $f^{-1}(x)$.

iv. Sketch the graph of $y = f^{-1}(x)$ on the axes below. Label all key features.



v. Show that, when the button width is halved the time taken by Minnie is increased by $\log_e \sqrt{2}$ seconds.

b) Mickey decides to find the values of a and b for his use. He finds that when $x = 1$, his time is 0.5 seconds, and when $x = 1.5$ his time is 0.3 seconds.

Question 8

Alex decides to investigate the sum of two functions, each with the same period. He considers the following:
$g(x) = f_1(x) + f_2(x)$, where $f_1(x) = \sqrt{3}\sin\left(\frac{\pi x}{3}\right)$ and $f_2(x) = \cos\left(\frac{\pi x}{3}\right)$, $\quad x \in [0, 12]$

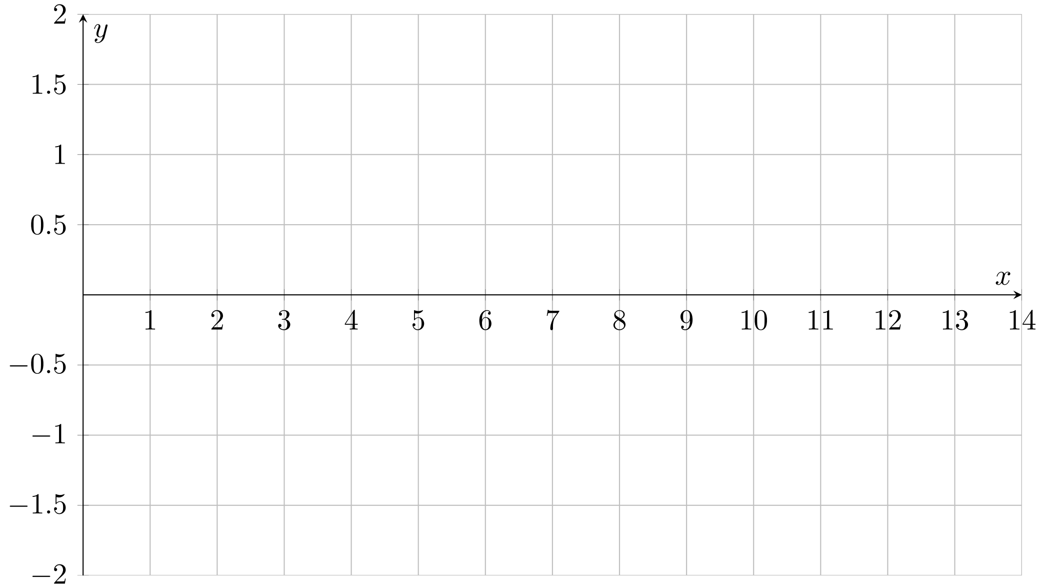
a) Calculate that y intercept on the graph of $g(x)$.

When he sketches the graph of $g(x)$, he notes that sum function is simply another circular function in the form of $y = A\sin\left(\frac{\pi}{3}x + \alpha\right)$. Where A > 0 and $\alpha \in \left[0, \frac{\pi}{2}\right]$

b) Given that $\sqrt{3}\sin\left(\frac{\pi x}{3}\right) + \cos\left(\frac{\pi x}{3}\right) = A\sin\left(\frac{\pi}{3}x + \alpha\right)$ Use the formula $\sin(A + B) = \sin(A)\cos(B) + \cos(B)\sin(A)$ to show that $A = 2$ and $\alpha = \frac{\pi}{6}$.

c) Using algebra, find x intercepts for $y = 2\sin\left(\frac{\pi x}{3} + \frac{\pi}{6}\right)$, where $x \in [0, 12]$.

d) On the axes provided, sketch $f_1(x), f_2(x)$ and $y = 2\sin\left(\dfrac{\pi x}{3} + \dfrac{\pi}{6}\right)$.



Question 9

For a particular sequence of transformations: $x \to -\dfrac{1}{2}x$, and $y \to \dfrac{1}{3}(y-2)$

a) State in the correct order, the sequence of transformations that point has undergone.

b) For the curve with equation $f(x) = \sqrt{x+2}$, state the new equation after it has undergone the transformations listed above.

c) A second function is defined by $g(x) = (x-1)^2 - 2$.

i. State the rule, of the composite function $f \circ g$.

ii. Explain, with reasoning, whether $f \circ g$.

iii. State the domain of $f \circ g$.

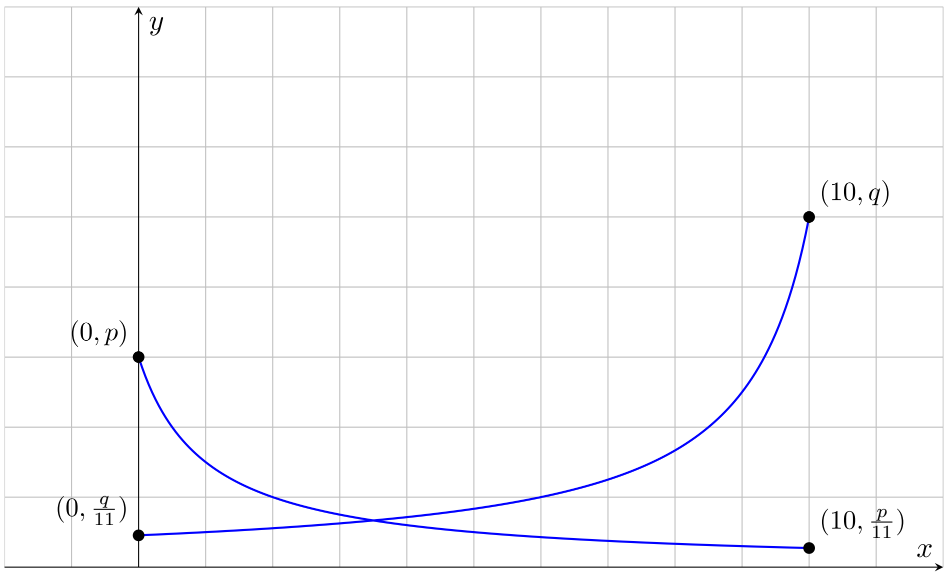
Question 10

The pollution level, y units along a straight road between factories A and B that are 10 km apart, is given by: $y = \dfrac{p}{x+1} + \dfrac{q}{11-x}$, where $0 \leq x \leq 10$

Where x km is the distance from factory A, and p and q are positive constants.

a) Find the pollution levels, y, in terms of p and q 3km along the road from A.

On March 21, the values of p and q are such that the two sections of the graph: $y_1 = \dfrac{p}{x+1}$ and $y_2 = \dfrac{q}{11-x}$ are shown below.



b) On the graph above, sketch the graph for pollution level on March 21, $y = y_1 + y_2$, labelling its two endpoints in terms of p and q.

c) Find in terms of p and q the coordinate of intersection between y_1 and y_2.

d) Hence find (also in terms of p and q) the value of pollution on March 21 at the point on the road with x value the same as that in **part c.**

Question 11

Consider the functions with rules $f(x) = \log_e(x)$ and $g(x) = -\log_e(3x+2)$.

a) State the implied domain of g.

b) List a sequence of transformations that maps the graph of $f(x)$ to the graph of $g(x)$.

c) Find the coordinates of the point(s) where the graph of $y = f(x)$ intersects the graph of $y = g(x)$.

d) Find $g^{-1}(x)$, the rule of the inverse of g.

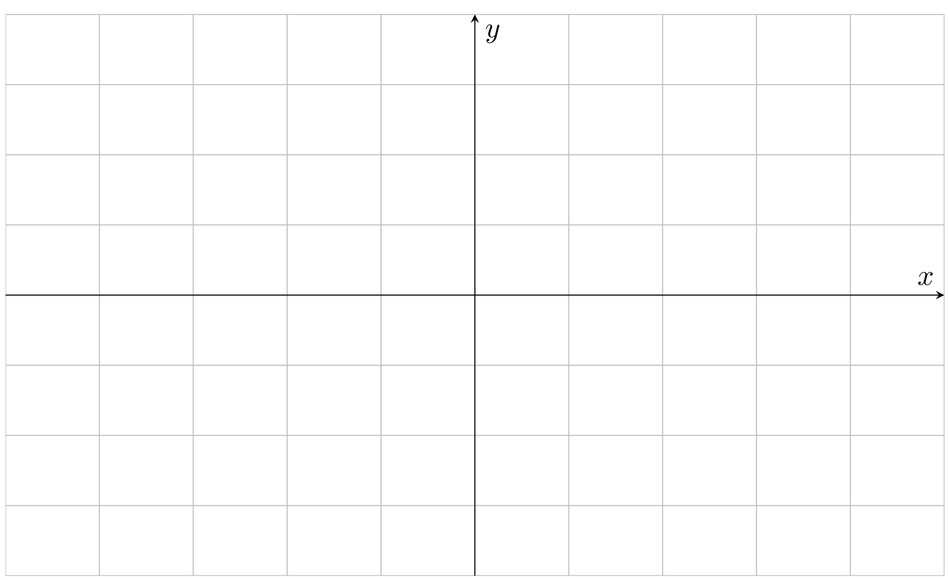
Question 12

Let $f : \mathbb{R} \to \mathbb{R}, f(x) = e^{\frac{x}{2}} - 1$

a) Find the rule and domain of the inverse function f^{-1}

b) Find $f(-f^{-1}(x))$ in the form $\dfrac{ax}{bx+c}$ where a, b and c are real constants.

c) On the axes provided below sketch the graph of $f(-f^{-1}(x))$ for its maximal domain.

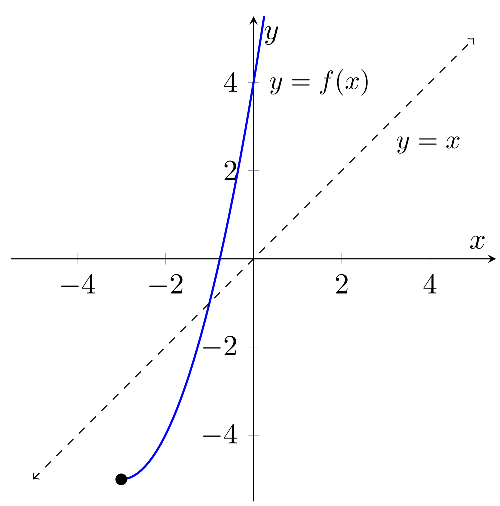


Question 13

$h(x) = (x+3)^2 - 5$ is the rule of a function with one possible inverse function described by:
$h^{-1} : S \to \mathbb{R}, h^{-1}(x) = \sqrt{x+5} - 3$

a) Restrict $h : [a, b) \to \mathbb{R}, h(x) = (x+3)^2 - 5$ and state the value of a and b, so that the inverse function takes the largest possible domain.

b) The graph of the restricted function h is shown opposite. On the same set of axes sketch $h^{-1}(x)$, labelling all key features.



c) Find the point of intersection of the graphs of $h(x)$ and $h^{-1}(x)$.