1. Let F_n be the n^{th} Fibonacci number where n is a positive integer. You are given that $F_1 = 1,\ F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for any $n \geq 3$. Prove using mathematical induction that F_{3a} is even for all positive integers a, i.e. $F_3, F_6, F_9, \cdots$ are even. (You may use the fact that $2c$ is even for any integer c, and that the sum of two even integers is even.)

2. Let $\{u_n\}$ be an arithmetic sequence with common difference d. Prove using mathematical induction that $u_1 + u_2 + \cdots + u_n = \frac{n}{2}(u_1 + u_n)$.

3. Use mathematical induction to prove that $\frac{d}{dx}(-\cos^n(x)) = \frac{n}{2}\sin(2x)\cos^{n-2}(x)$ for all positive integers n.

4. Use mathematical induction to prove that the sum of n terms of a finite geometric sequence is $\frac{u_1(1-r^n)}{1-r}$, where u_1 is the first term of the sequence and r is the common multiplier.

5. Prove, by mathematical induction, that

$$\cos(2x)\cos(2^2x)\cos(2^3x)\cdots\cos(2^nx) = \frac{\sin(2^{n+1}x)}{2^n\sin(2x)}$$

for all positive integers n.

6. Let $R_2 = e^x$, $R_3 = x^{e^x}$, $R_4 = e^{x^{e^x}}$ etc., i.e. let R_n be the iteration of this pattern which has $n-1$ exponents for any integer $n \geq 2$. If n is even, the base is e; if n is odd, the base is x.
   a. [6] Show that $\frac{d}{dx}R_3 = R_3R_2\left(\frac{1}{x} + \ln x\right)$. *(Hint: let $y = R_3$ and begin by taking both sides to the natural logarithm.)*
   b. [2] Hence show that $\frac{d}{dx}R_4 = R_4R_3R_2\left(\frac{1}{x} + \ln x\right)$.
   c. [5] Using mathematical induction, prove that if n is even and $n \geq 4$, then

$$\frac{d}{dx}R_n = R_n\left(\frac{d}{dx}R_{n-1}\right)$$

   (Hint: for the inductive step, show that $P(k+2)$ is true.)

   d. [8] Using mathematical induction, prove that if n is odd and $n \geq 3$, then

$$\frac{d}{dx}R_n = R_nR_{n-1}\left(\frac{1}{x} + R_{n-2}R_{n-3}\ln x\left(\frac{1}{x} + R_{n-4}R_{n-5}\ln x\cdots\left(\frac{1}{x} + R_3R_2\ln x\left(\frac{1}{x} + \ln x\right)\right)\right)\right).$$