THE INSTITUTE FOR ENQUIRING MINDS

redefining
access
resetting
attitudes

enquiringminds.org
info@enquiringminds.org
+61 448 218652

Specialist Maths Units 3/4

Vector Functions

Practice Questions

Short Answer Questions

Question 1

The position of an object is $r(t)$ metres at time t where

$$r(t) = \tan\left(\frac{\pi t}{10}\right)\underset{\sim}{i}\,, \quad t \in [0, 5)$$

Find:

a) The velocity vector.

b) The speed when $t = 2.5$

c) The average velocity in the interval [0, 2.5]

Question 2

The position vector, $r(t)$, of a particle at time t is given by $r(t) = (e^{-t}\sin(t))\underset{\sim}{i} + (e^{-t}\cos(t))\underset{\sim}{j}$. Show that the speed of the particle at time t is ke^{-t}, where k is a constant that should be determined.

Question 3

Two toy boats start at a common point (0,0) at $t = 0$ and move with velocity vectors:

$$v_1(t) = t\underset{\sim}{i} + (t^2 - t)\underset{\sim}{j}$$

$$v_2(t) = -t\underset{\sim}{i} + (t^2 + t)\underset{\sim}{j}$$

Distance is measured in metres, time is measured in seconds.

a) What is the angle between their respective directions of motion at $t = 2.0$?

b) How far apart are they at $t = 2.0$?

c) At what time, if any, are the acceleration vectors $a_1(t)$ and $a_2(t)$ perpendicular to each other?

Question 4

Two objects, 1 and 2 in an $X-Y$ plane with position vectors $r_1(t) = (t+1)\underset{\sim}{i} + \dfrac{12}{t+1}\underset{\sim}{j}$ and $r_2(t) = 6\sin(at)\underset{\sim}{i} + 6\cos at\underset{\sim}{j}$ with $t \geq 0$ respectively and a as a positive real number. Take the unit vector $\underset{\sim}{i}$ to be parallel to the x-axis and the unit vector $\underset{\sim}{j}$ to be perpendicular to the x-axis. Length is measured in metres and time in seconds.

a) Given the Cartesian equation in terms of x and y for their paths along with an appropriate specification of domain and range for each path.

b) Describe the motion of both objects.

c) The paths of both objects intersect at points P and Q with object 1 passing through P first. Find and state the co-ordinates of P and Q.

d) Find the least value of a such that the two objects collide at the point Q as an exact value.

Question 5

The acceleration of a particle moving in the $x - y$ plane is $-g\underset{\sim}{j}$, At time $t = 0$, the particle leaves the point with position $\underset{\sim}{r}(0) = h\underset{\sim}{j}$ with velocity $\underset{\sim}{v}(0) = V\cos(\theta)\underset{\sim}{i} + V\sin(\theta)\underset{\sim}{j}$.

a) Show that the particle's position vector at time t is given by

$$\underset{\sim}{r}(t) = V\cos(\theta)t\underset{\sim}{i} + \left(V\sin(\theta)t - \frac{1}{2}gt^2 + h\right)\underset{\sim}{j}.$$

b) Show that the particle's path is given by $y = h + \tan(\theta)x - \dfrac{g\sec^2(\theta)}{2V^2}x^2$.

When a projectile is fired horizontally with a speed of U m/s from the top of a cliff of height h metres above sea level, the projectile hits a stationary target in the water. In addition, if the projectile is fired from the same position with a speed of U m/s, but with an angle of elevation of $\tan^{-1}(3)$, the projectile also hits the target.

c) Find, in terms of U and g, an expression for the horizontal distance, x metres travelled by the projectile.

Multiple Choice Questions

Question 1

A particle moves in a straight line so that its displacement, x m, from a fixed point at any time t seconds is given by $x = 2\sin(t)\underset{\sim}{i} + 3\cos(t)\underset{\sim}{j}$. The first time, in seconds, when the particle is at rest is closest to:

A. 0.59

B. 1.02

C. 0.98

D. 33.7

E. 1.70

Question 2

At time t, the position vector of a particle with coordinates (x, y) is given by $r(t) = \sqrt{t}\underset{\sim}{i} + \frac{1}{t+1}\underset{\sim}{j}$. This particle moves along the path with equation:

A. $\sqrt{x}(y+1) = 1$

B. $x^2(y-1) = 0$

C. $y = x^2 - 1$

D. $y = \dfrac{x^2-1}{x^2}$

E. $y(x^2+1) = 1$

Question 3

Let $\underset{\sim}{r} = 2\sin(2t)\underset{\sim}{i} + 3\cos(2t)\underset{\sim}{j}$ be a time dependent position vector of an object. The path of the object takes is:

A. A straight line

B. A parabola

C. A circle

D. An ellipse

E. A hyperbola

Question 4

The velocity of a particle is given by $\underset{\sim}{v} = -2\cos(2t)\underset{\sim}{i} + \underset{\sim}{j}$ m/s, $t \geq 0$ seconds.
The acceleration when the particle first has a speed of $\sqrt{3}$ m/s is:

A. $\sqrt{2}\underset{\sim}{i} + \underset{\sim}{j}$

B. $-2\sqrt{2}\underset{\sim}{i}$

C. $2\sqrt{3}\underset{\sim}{i}$

D. $-2\underset{\sim}{i}$

E. $2\sqrt{2}\underset{\sim}{i}$

Question 5

A projectile is launched from the point $r(0) = 10\underset{\sim}{j}$ such that the velocity vector is given by $v(t) = 20\underset{\sim}{i} + (30 - 9.8)\underset{\sim}{j}$ m/s. The position vector of the projectile is:

A. $\underset{\sim}{r} = 9.8\underset{\sim}{j}$

B. $\underset{\sim}{r} = (20t + 10)\underset{\sim}{i} + (30t - 9.8t^2)\underset{\sim}{j}$

C. $\underset{\sim}{r} = 20t\underset{\sim}{i} + (20t - 4.9t^2)\underset{\sim}{j}$

D. $\underset{\sim}{r} = 20t\underset{\sim}{i} + (30t - 4.9t^2 + 10)\underset{\sim}{j}$

E. $\underset{\sim}{r} = (20t - 10)\underset{\sim}{i} + (30t - 4.9t^2)\underset{\sim}{j}$

Question 6

A space shuttle returns to Earth and approaches the runaway with a velocity $\underset{\sim}{v} = 60\underset{\sim}{i} - 80\underset{\sim}{j} - 8\underset{\sim}{k}$. It is observed on a control tower radar screen at time $t = 0$ seconds. Five seconds later the shuttle passes over a navigation beacon with position vector $-600\underset{\sim}{i} + 2400\underset{\sim}{j}$ relative to the base of the control tower, at an altitude of 220 metres. The angle from the runway the space shuttle lands, correct to the nearest degree, is

A. $8°$

B. $7°$

C. $6°$

D. $5°$

E. $4°$

Question 7

A particle has a path defined by $r(t) = 3\sec(t)\underset{\sim}{i} - 4\tan(t)\underset{\sim}{j}, \quad t \in \left[\frac{7\pi}{4}, 2\pi\right]$. The rule describing the path on Cartesian axes is:

A. $x^2 = \frac{3}{4}y^2 = 1, x \in [3, 3\sqrt{2}]$

B. $\frac{x^2}{9} + \frac{y^2}{4} = 1, x \in [3, 3\sqrt{2}], y \in [0, 4]$

C. $\frac{x^2}{9} - \frac{y^2}{16} = 1, x \in [3, 3\sqrt{2}], y \in [0, 4]$

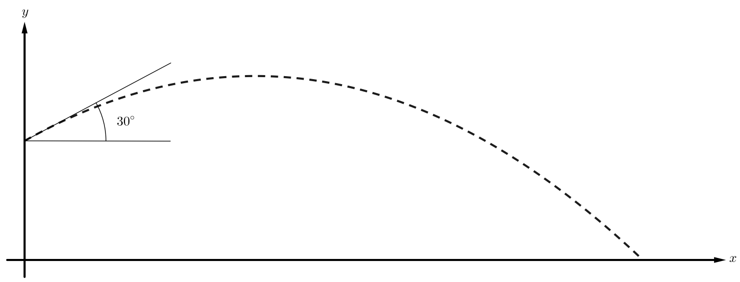
D. $\frac{x^2}{9} - \frac{y^2}{16} = 1, x \in [3, 3\sqrt{2}], y \in [-4, 0]$

E. $16x^2 - 9y^2 = 144, x \in [0, 3], y \in [0, 4]$

Extended Response Questions

Question 1

A bowling machine is programmed to project a ball from a height of 1.5m above the ground at an angle of 30° to the horizontal as shown below. The ball's initial speed is 8ms^{-1} and it travels in a vertical plane, landing on the ground T seconds later.
On a Cartesian graph, the origin O is located 1.5m vertically below the point where the ball is projected. Let $\underset{\sim}{i}$ represented a unit vector in the positive x direction and let $\underset{\sim}{j}$ represent a unit vector in the positive y direction where displacement is measured in metres and time is in seconds.



a) Show that the ball has an initial velocity of $4\sqrt{3}\underset{\sim}{i} + 4\underset{\sim}{j}$.

b) The acceleration of the ball t seconds after being projected is given by:

$$\underset{\sim}{a}(t) = -\frac{t}{50}\underset{\sim}{i} + \left(\frac{t}{20} - g\right)\underset{\sim}{j}, 0 \leq t \leq T$$

Show that the position vector of the ball t seconds after begin projected by the machine is given by

$$\underset{\sim}{r}(t) = \left(4\sqrt{3}t - \frac{t^3}{300}\right)\underset{\sim}{i} + \left(\frac{t^3}{120} - \frac{gt^2}{2} + 4t + 1.5\right)\underset{\sim}{j}, 0 \leq t \leq T$$

c) How far does the ball travel horizontally before it hits the ground? Give your answer in metres correct to 1 decimal place.

d) What is the distance, in metres, that the ball travels from when it is projected until it hits the ground? Give your answer to one decimal place.

e) Find the speed of the ball (in m/s) when it hits the ground/ Give your answer correct to one decimal place.

Question 2

A young Elon Musk fires a model rocket from his backyard where the unit vector $\underset{\sim}{i}$ points east, the unit vector $\underset{\sim}{j}$ points north and the unit vector $\underset{\sim}{k}$ points vertically up. The rocked is projected so that its velocity $v(t)$ m/s at time t seconds $(0 \leq t \leq 3)$ is given by

$$v(t) = 6\underset{\sim}{i} + 2\sqrt{3}\underset{\sim}{j} + (59.6t - 14.9t^2)\underset{\sim}{k}$$

a) Find the velocity of the rocket at $t = 3$ and hence find its speed at $t = 3/$

b) Find an expression for the acceleration of the rocket at time $t, (0 < t < 3)$.

c) Show that the rocket reaches a height of 134 metres at $t = 3$.

d) For $t > 3$, the rocket is subject only to acceleration due to gravity, that is -9.8m/s^2. Find the maximum height reached by the rocket, correct to the nearest metre.

e) Find the position vector of the rocket when it is a maximum height.

Question 3

A javelin is thrown by a competitor on level ground. At time t, the time in seconds measured from the release of the javelin, the position vector $\underset{\sim}{r}(t)$ of the tip of the javelin is given by

$$\underset{\sim}{r}(t) = 34t\underset{\sim}{i} + \left(\frac{\pi}{2}t - 4\cos\left(\frac{\pi t}{6}\right)\right)\underset{\sim}{j} + (5 + 34t - 7t^2)\underset{\sim}{k}$$

Where $\underset{\sim}{i}$ is a unit vector in the east direction, $\underset{\sim}{j}$ is a unit vector in the north direction and $\underset{\sim}{k}$ is a unit vector vertically up. The origin O of the coordinate system at ground level and displacements are measured in metres.

Let P be the point where the tip of the javelin hits the ground.

a) Show that the tip of the javelin reaches P in 5 seconds

b) How far from O, correct to the nearest centimetre, does the tip of the javelin hit the ground?

c) **i.** Find the velocity vector of the tip of the javelin at time $t = 5$.

ii. Find the speed of the tip of the javelin at time $t = 5$. (Answer to one decimal place)

iii. Find the position of the tip of the javelin at time $t = 5$.

iv. At what angle, correct to the nearest tenth of a degree, does the path of the javelin's tip make with the ground at P?

Question 4

A particle moves so that its position vector at time t, $t \geq 1$, is given by

$$\underset{\sim}{r}(t) = \left(t + \frac{1}{t}\right)\underset{\sim}{i} + \left(t - \frac{1}{t}\right)\underset{\sim}{j}$$

a) Find an expression for the distance from the origin of the particle at time t.

b) Find the speed of the particle at time $t = 5$.

c) **i.** Find the Cartesian equation of the path of the particle and state the domain and range

ii. Sketch the path of the particle.

A second particle has a velocity vector given by

$$\underset{\sim}{v_B}(t) = \left(2 - \frac{2}{t^2}\right)\underset{\sim}{i} + \left(2 + \frac{2}{t^2}\right)\underset{\sim}{j}, t \geq 0$$

When $t = 1$, the position vector is given by $\underset{\sim}{r_B}(t) = 4\underset{\sim}{i}$.

d) Show that the position vector of this second particle and of the first particle at time t are parallel.

Question 5

A stone is thrown from an initial position of $\underset{\sim}{r}(0) = 0$ to hit a small object on the top of a wall which is 20 metres horizontally from the point of projection and 5 metres high.

a) If the stone which has an acceleration of $\underset{\sim}{a} = -g\underset{\sim}{j}$ is thrown from the horizontal ground with a speed of 40ms^{-1} at an angle of θ from the horizontal, show that the equation of the path of the stone is given by

$$\underset{\sim}{r}(t) = 40\cos(\theta)t\underset{\sim}{i} + (40\sin(\theta) - \frac{1}{2}gt^2)\underset{\sim}{j}.$$

b) Hence, show that two possible angles of elevation, θ, of the stone to hit the object are $86.43°$ and $17.61°$.

Question 6

The position of a particle at any time t is given by:

$$\underset{\sim}{r}(t) = (2\cos(t) + 1)\underset{\sim}{i} + (\sin(2t) + \sin(t))\underset{\sim}{j}, t \geq 0$$

a) Find the velocity of the particle at $t = \dfrac{\pi}{6}$.

b) Find the value of k for which $3k\underset{\sim}{i} + 4\underset{\sim}{j}$ is perpendicular to $\dfrac{d\underset{\sim}{r}}{dt}$ at $t = \dfrac{\pi}{6}$.

c) Show that if $\underset{\sim}{r}(t) = x\underset{\sim}{i} + y\underset{\sim}{j}$ then $\sin(t) = \dfrac{y}{x}$ and $\cos(t) = \dfrac{1}{2}(x - 1)$.

d) Hence show that the Cartesian equation of the path is $4y^2 = 3x^2 + 2x^3 - x^4$.

Question 7

Two particles are moving in a 3D-space. Both particles have non-intersecting straight line paths. The position vector of particle A is given by

$$\underset{\sim}{r}(t) = \underset{\sim}{i} - 2\underset{\sim}{j} + t(2\underset{\sim}{i} + 3\underset{\sim}{j} - 4\underset{\sim}{k})$$

where the vector $2\underset{\sim}{i} + 3\underset{\sim}{j} - 4\underset{\sim}{k}$ is parallel to the path of A.
The position vector of particle B is given by

$$\underset{\sim}{s}(t) = \underset{\sim}{j} + t(3\underset{\sim}{i} + \underset{\sim}{j} - 5\underset{\sim}{k})$$

where the vector $3\underset{\sim}{i} + \underset{\sim}{j} - 5\underset{\sim}{k}$ is parallel to the path of B.

Take all distances to be in metres and t is time, measured in seconds.

a) The vector $\underset{\sim}{m} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$ where $x > 0$ is unit vector perpendicular to both $2\underset{\sim}{i} + 3\underset{\sim}{j} - 4\underset{\sim}{k}$ and $3\underset{\sim}{i} + \underset{\sim}{j} - 5\underset{\sim}{k}$. Find the value of x, y and z.

b) Find the coordinates of the positions for both particles at $t = 0$.

The shortest distance between two non-parallel lines $\underset{\sim}{L_1} = P + t\underset{\sim}{u}$ and $\underset{\sim}{L_2} = Q + s\underset{\sim}{v}$ is

$$D = \left|(Q - P) \cdot \underset{\sim}{m}\right|$$

c) Use the formula above the find the minimum distance between the path of particle A and the path of particle B.

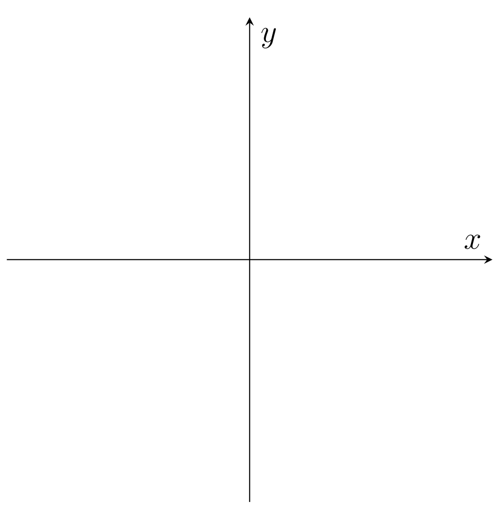
Question 8

A basketball player shoots the ball from 2m above the ground towards a basket that is 3m above the ground and 5m away in the horizontal direction. Assume the unit vector $\underset{\sim}{i}$ points in the direction of increasing x, unit vector $\underset{\sim}{j}$ points in the direction of increasing y, and the shooter is standing at $-5\underset{\sim}{i}$ when he takes the shot.

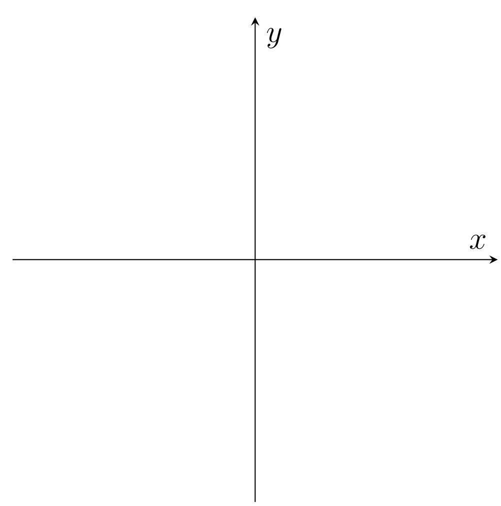
a) Suppose the player shoots the ball at an angle of a above the horizontal, where $0 \leq a \leq \frac{\pi}{2}$ with an initial speed V. Find the vector expression for the displacement of the ball at time t seconds, $\underset{\sim}{r}(t)$.

b) Using your result from **part a.**, plot the path the ball takes for:

i. $V = 7, a = \frac{\pi}{4}$



ii. $V = 8, a = \frac{\pi}{3}$



c) Find the angle $A(0 < A < \frac{pi}{2})$ that minimises velocity needed for the ball to go into the basket, and find this minimum velocity.

[Hint: Start by finding an equation in V and a that holds if the enters the hoop]

Question 10

The motion of two particles is described by $\underset{\sim}{r_1}(t) = t\underset{\sim}{i} + t^2\underset{\sim}{j} + t^3\underset{\sim}{k}$ and $\underset{\sim}{r_2} = (1+2t)\underset{\sim}{i} + (1+6t)\underset{\sim}{j} + (1+14k)\underset{\sim}{k}$ where $t \geq 0$.

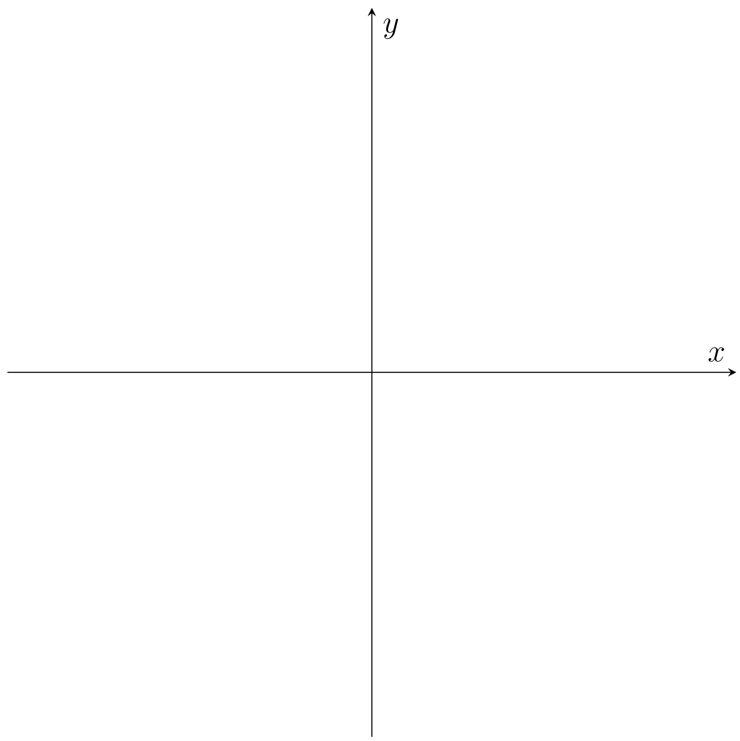
a) Show that the two particles do not collide

b) Find any points at which the two paths intersect.

c) The motion of two different particles is described by $\underset{\sim}{r_1} = \underset{\sim}{i} + (5t+4p)\underset{\sim}{j} + 3\underset{\sim}{k}$ and $\underset{\sim}{r_2}(t) = (4t+p)\underset{\sim}{i} + (7t+q)\underset{\sim}{j} + (t-q)\underset{\sim}{k}$. where $t \geq 0$ and p and q are real constants.

i. Find a pair of values for p and q such that the particles collide, and find the time and place of collision.

ii. Sketch the paths of the two particles for $0 \leq t \leq 1$ as projected onto the sy-plane, that is, as would be seen by an observer watching from directly above. Label the point of collision and the end-points of each path.



Question 11

A football is kicked in a vertical plane and has position vector given by $\underset{\sim}{r}(t) = x(t)\underset{\sim}{i} + y(t)\underset{\sim}{j}$ for $y \geq 0$ and $0 \leq t \leq T$. Where t is the time in seconds, $\underset{\sim}{i}$ is a unit vector of one metre horizontally forward and $\underset{\sim}{j}$ is unit vector one metre vertically upwards, above ground level. Initially the football is kicked from a point one metre above the ground, and is kicked into headwind, so that its velocity vector is given by

$$\underset{\sim}{v}(t) = (12 - 0.1x(t))\underset{\sim}{i} + (12 - gt)\underset{\sim}{j}.$$

a) Using integration, show that $x(t) = 120(1 - e^{-\frac{1}{10}t})$

b) Show that $y(t) = 1 + 12t - 4.9t^2$.

c) Find the time T, in seconds, when the football hits the ground. Give your answer correct to three decimal places.

d) Find the range of the football in metres. Give your answer correct to three decimal places.

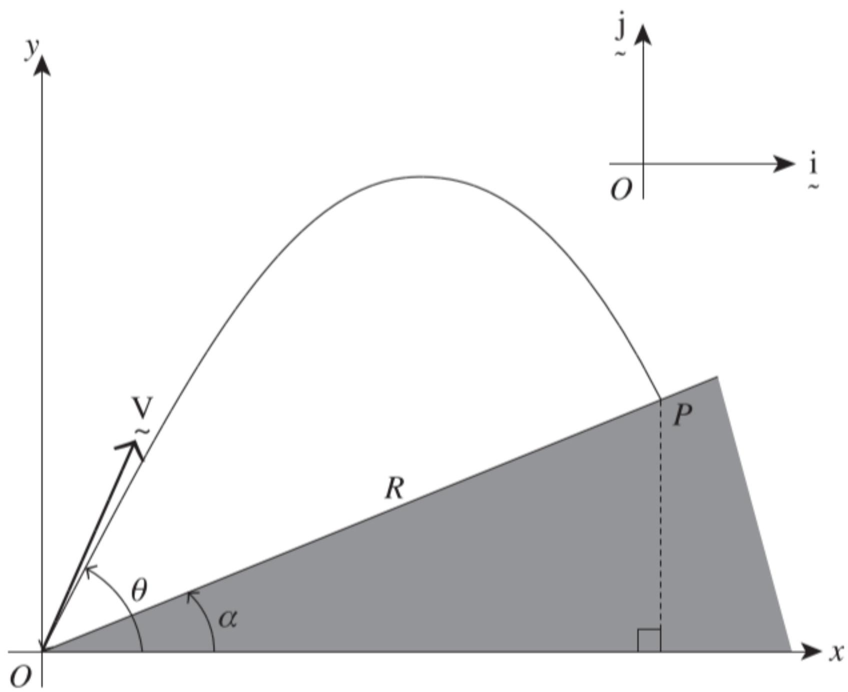
e) Find the time in seconds when the football reaches its maximum height. Find the maximum height in metres reached and the horizontal distance travelled in metres at this time. Give all answers correct to three decimal places.

f) Determine the speed and angle at which the football hits the ground. Give your answer for the speed in ms^{-1} correct to three decimal places, and the angle in degrees and minutes.

Question 12

The diagram below shows an inclined plane that makes an angle α with the horizontal. A projectile is foxed from O at the bottom of the incline with velocity $v(t) = V\cos(\theta)\underset{\sim}{i} + V\sin(\theta)\underset{\sim}{j}$ m/s at an angle θ degrees to the horizontal, where $0° < \theta < 90°$.

The projectile hits the inclined plane at P. The acceleration of the projectile moving in the xy-plane is $-g\underset{\sim}{j}$. Let (x, y) be the projectile's position after t seconds of flight.



The Cartesian equation describing the projectile's path is

$$y = \tan(\theta)x - \frac{g}{2V^2}(1 + \tan^2(\theta))x^2.$$

a) Show that OP lies on the line with equation $y = \tan(\alpha)x$.

b) Show that the projectile's range R metres up the inclined plane is given by

$$R = \frac{2V^2 \sin(\theta - \alpha)\cos(\theta)}{g\cos^2(\alpha)}$$

c) **i.** Find an expression for θ such that $\dfrac{dR}{d\theta} = 0$.

ii. Find $\dfrac{d^2R}{d\theta^2}$ and hence verify that the expression for θ found in **part c.i.** gives the projectile's maximum range up the inclined plane.

d) For the trajectory of maximum range, use the result that $\dfrac{dy}{dx} = \dfrac{-(\cos(2\theta)+1)}{\sin(2\theta)}$ to show that the initial direction of the particle's trajectory is perpendicular to the direction at which the projectile hits the inclined plane.